Smoothed Asymptotics for Divergent Series

To evaluate a series, typical way is to calculate the N^{th} partial sum and then take the limit as $N \to \infty$:

$$\sum_{n=1}^{\infty} f(n) \to \sum_{n=1}^{N} f(n) = \sum_{n=1}^{\infty} f(n)\, \mathbf{1}_{[0,1]}(n/N) \ . \tag{1}$$

When this limit doesn't exist, we say the series is divergent.

Regularization generalizes this approach. Choose a *regulator* $\eta(x)$ such that $\eta(0) = 1$ and $\eta(x) \to 0$ as $x \to \infty$. Calculate the N^{th} "smoothed" sum:

$$\sum_{n=1}^{\infty} f(n) \to \sum_{n=1}^{\infty} f(n)\, \eta(n/N) \ . \tag{2}$$

Again, try to take the $N \to \infty$ limit to calculate a value for the series.

If the regulator is a sufficiently smooth function, divergences are tamed and the sum has well-defined asymptotic behavior as $N \to \infty$.

Partial sums are not smooth at all - they correspond to a regulator that is not continuous!

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A divergent series, when regularized smoothly enough, takes the form:

$$\sum_{n=1}^{\infty} f(n)\eta(n/N) = (c_{\eta,1}N + \ldots + c_{\eta,s}N^s) + \alpha + \left(\frac{d_{\eta,1}}{N} + \frac{d_{\eta,2}}{N^2} + \ldots\right) , \qquad (3)$$

where the N-independent term α is also independent of the choice of regulator!

Terms polynomial in N diverge as $N \to \infty$, but there is still a universal finite piece of the series that can be unambiguously extracted via regularization.

- **Riemann-Zeta function** $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ can be extended to negative integers. After regularization, the constant term is the same one you would obtain through analytic continuation on the complex plane.
- **Casimir effect** in quantum field theory predicts the energy of a field between two large conducting plates:

$$E = \frac{\hbar}{2} \sum_{n=1}^{\infty} \omega_n \ , \quad \omega_n = \frac{n\pi c}{L} \ . \qquad (4)$$

After regularization, the finite piece can be calculated. Experimental results match this prediction!