Exercise 1

Grandi's series is a particularly simple divergent series:

$$\sum_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + \dots \; . \tag{1}$$

Calculate the N^{th} partial sum $S_N = \sum_{n=1}^{N}(-1)^{n+1}$ of Grandi's series. What can you say about the $N \to \infty$ limit?

Exercise 2

Let's ignore the fact that Grandi's series is divergent and attempt to assign a numerical value to it anyways:

$$S := 1 - 1 + 1 - 1 + \dots \; . \tag{2}$$

Intuitively, is there a value that would make sense for S to take? Try to justify this answer by expanding out the quantity $1 - S$ and then rewriting that in terms of S again.

Exercise 3

Let's try regularizing Grandi's series with the following regulator:

$$\eta(x) = (1 - x)_+ := \max(1 - x, 0) \; . \tag{3}$$

This choice of regulator corresponds to *Cesàro summation*, where a series is assigned a value equal to the limit, as $m \to \infty$, of the arithmetic mean of the series' first m partial sums.

Using this regulator, calculate the N^{th} smoothed sum S_N of Grandi's series, i.e.

$$S_N := \sum_{n=1}^{\infty}(-1)^{n+1}\eta\left(\frac{n}{N}\right) \; . \tag{4}$$

Compare and contrast your results with the results from Exercises 1 and 2. What matches, and what doesn't?

You may find the following identity useful:

$$\sum_{n=1}^{N}(-1)^{n+1}n = \begin{cases} -\frac{N}{2} & N \text{ even} \\ \frac{N+1}{2} & N \text{ odd} \end{cases} \tag{5}$$

Exercise 4

Consider the following series, which we'll refer to as the *all-ones series*:

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots \; . \tag{6}$$

Calculate the N^{th} partial sum $T_N := \sum_{n=1}^{N} 1$ of this series. What happens in the limit where $N \to \infty$?

Exercise 5

Let's proceed in the same way as Exercise 2 and pretend that Grandi's series and the all-ones series have finite values:

$$\begin{aligned} S &= 1 - 1 + 1 - 1 + \dots \ , \\ T &= 1 + 1 + 1 + 1 + \dots \ . \end{aligned} \tag{7}$$

Expand out the quantity $S - T$ and try to rewrite it in terms of S. Plug in the value of S that you got from exercise 2, and solve for T. Notice anything odd about the result?

Exercise 6

Using the Cesàro regulator (3), calculate the N^{th} smoothed sum T_N of the all-ones series:

$$T_N := \sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) \ . \tag{8}$$

Compare and contrast with your results Exercises 4 and 5. What matches, and what doesn't?

You may find the following identity useful:

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \tag{9}$$

Exercise 7

A common choice of regulator is the exponential regulator $\eta(x) = e^{-x}$. Use this regulator to calculate the N^{th} smoothed sum of both Grandi's series and the all-ones series, i.e.

$$S_N := \sum_{n=1}^{\infty} (-1)^{n+1} \eta\left(\frac{n}{N}\right) \ , \quad T_N := \sum_{n=1}^{\infty} \eta\left(\frac{n}{N}\right) \ . \tag{10}$$

What happens to your results as $N \to \infty$?

Hint: recall that, for any $r \in \mathbb{R}$ with $|r| < 1$, its geometric series is given by

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \ . \tag{11}$$

Exercise 8

Instead of immediately taking the $N \to \infty$ limit, let's study the asymptotics by leaving N finite but large. Write down a series expansion, in powers of N, for the N^{th} smoothed sums of both series, as computed in the previous exercise.

In these series expansions, which terms survive as $N \to \infty$? Do you notice any similarities between these expansions and your answers from Exercises 3 and 6?

You may find the following Taylor series useful:

$$\begin{aligned} \frac{1}{1+e^{-x}} &= \frac{1}{2} + \frac{x}{4} + \mathcal{O}(x^3) \ , \\ \frac{x}{1-e^{-x}} &= 1 + \frac{x}{2} + \frac{x^2}{12} + \mathcal{O}(x^3) \ . \end{aligned} \tag{12}$$

Bonus Exercises (hard)

Exercise 9

Consider the following divergent series:

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \dots \ . \tag{13}$$

Calculate the N^{th} smoothed sum of this series using the exponential regulator.

Then, holding $N \gg 1$ to be large but finite, Taylor expand the result in powers of N, dropping any terms that are $\mathcal{O}(1/N)$ or subleading.

Hint: if you need to calculate a series of the form $\sum_{n=0}^{\infty} nr^n$, try differentiating the geometric series $\sum_{n=0}^{\infty} r^n$ with respect to r.

Exercise 10

Calculate the smoothed version of the series $1 + 2 + 3 + 4 + \dots$ using the following regulator[a]:

$$\eta(x) = \max\left((1-x)^2, 0\right) \ . \tag{14}$$

Compare and contrast with the previous exercise. What's the same? What's different?

[a] Note that this is the square of the Cesàro regulator from Exercise 6. Squaring helps smooth the regulator out at the transition point $x = 1$. (i.e. since our series is more divergent than previous ones we looked at, the regulator must also be smoother.) We didn't have to worry about this with the exponential regulator, because it is already C^{∞} everywhere.

Exercise 11 – Regulator independence for Grandi's series

Suppose that $\eta : \mathbb{R}^+ \to \mathbb{R}$ is a regulator with the following properties:

- $\eta(0) = 1$.
- $\eta(x) \to 0$ smoothly as $x \to \infty$.
- η is bounded on $\mathbb{R}^+$.
- η is at least twice-differentiable everywhere on $\mathbb{R}^+$, i.e. $\eta \in C^2$.
- η has compact support on $\mathbb{R}^+$.

For any such η, prove that the regularization of Grandi's series takes the form

$$\sum_{n=1}^{\infty} (-1)^{n+1} \eta\left(\frac{n}{N}\right) = \frac{1}{2} + \mathcal{O}(1/N) \ . \tag{15}$$

1. First, show that the N^{th} smoothed sum can be written as

$$\frac{1}{2}\eta\left(\frac{1}{N}\right) + \frac{1}{2}\sum_{m=1}^{\infty}\left[\eta\left(\frac{2m-1}{N}\right) - 2\eta\left(\frac{2m}{N}\right) + \eta\left(\frac{2m+1}{N}\right)\right] \ . \tag{16}$$

2. Use the fact that $\eta \in C^2$ to perform a Taylor expansion to approximate this N^{th} smoothed sum, for $N \gg 1$, as

$$\frac{1}{2} + \frac{1}{2N^2}\sum_{m=1}^{\infty} \eta''\left(\frac{2m}{N}\right) \ . \tag{17}$$

3. Using the fact that η is bounded and has compact support, argue that for sufficiently large N, the summand is non-zero only for $m \lesssim N$, and the entire sum is at most $\mathcal{O}(N)$. Thus, we reproduce (15).

Importantly, the leading order term in the smoothed sum is regulator-independent and survives in the $N \to \infty$ limit. Note that smoothness of η was crucial in this proof!