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On-Shell Actions in Euclidean Supergravity

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Abstract: Goal of this note: explicitly spell out Euclidean $\mathcal{N} = 2$ supergravity and the Euclidean STU model, derive the on-shell actions for various solutions, and use them to bootstrap a general formula.

Contents

1 The big idea

The goal of this project is to find novel solutions to the STU model in four-dimensional Euclidean supergravity and to compute their corresponding on-shell actions. This has been explored so far in [1], where the conformal boundary is S^3, and in [2], where the conformal boundary is $S^1 \times \Sigma_{\mathfrak{g}}$. There are a few reasons that finding new solutions beyond these and computing their on-shell action would be worthwhile:

1. The space of Euclidean solutions in supergravity is much larger than that of Lorentzian solutions. The BPS conditions become less restrictive, due to the "doubling" that occurs in the continuation from Lorentzian to Euclidean signature (see Section 4), and solutions that are singular in Lorentzian signature may be smooth in Euclidean.

2. The STU model is a truncation of 4d $\mathcal{N}=8$ supergravity, which in and of itself is a consistent truncation of M-theory compactified on a 7d Sasaki-Einstein manifold X_7. By finding novel solutions that admit an uplift to M-theory, we can elaborate on the $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence and learn more about the dual ABJM theory. In particular, the Euclidean nature of our bulk means that we can learn about configurations of ABJM on compact, Euclidean manifolds that admit no Lorentzian continuation.

3. There may be a general formula for the on-shell action of BPS solutions in the STU model, along the lines of the formula derived in [3] for BPS solutions in minimal supergravity. Whatever this bulk on-shell action formula is, it must be consistent with equation (2.69) of [4] for the partition function of ABJM on the manifold $\mathcal{M}_{\mathfrak{g},p}$, defined as a $U(1)$ bundle of degree p over a Riemann surface $\Sigma_{\mathfrak{g}}$ of genus $\mathfrak{g}$. Such a relation would necessarily generalize the UV/IR correspondence noted in [2], where there is a precise relation between the scalars in the IR of the solution and the CFT sources, as read off from the UV region of the bulk solution, for ABJM on $S^1\times\Sigma_{\mathfrak{g}}$. There is also a UV/IR relationship present for ABJM on S^3, as detailed in Section 6.5, although this UV/IR correspondence was not explicitly detailed in [1].

2 Conventions

Throughout this note, we consider four-dimensional Lorentzian and Euclidean spacetimes, with metric signature $(-+++)$ and $(++++)$, respectively. We denote spacetime indices by Greek letters $\mu,\nu,\ldots$ and flat tangent space indices by Latin letters $a,b,\ldots$. In Lorentzian signature, the tangent space indices take values $(0,1,2,3)$, while in Euclidean signature they take values $(1,2,3,4)$.

Our Levi-Civita tensor conventions are

$$\varepsilon_{\mu\nu\rho\sigma}=\varepsilon_{abcd}e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d\ , \tag{2.1}$$

such that the tensor takes values $\pm e=\pm\sqrt{|g|}$. We also define the gamma matrix product γ_5 by

$$\gamma_5=\begin{cases} i\gamma_0\gamma_1\gamma_2\gamma_3 & \text{in Lorentzian signature}\ ,\\ \gamma_1\gamma_2\gamma_3\gamma_4 & \text{in Euclidean signature}\ ,\end{cases} \tag{2.2}$$

such that γ_5 anti-commutes with all gamma matrices and $\gamma_5^2=1$ in both signatures. For a $U(1)$ field strength $F_{\mu\nu}$, we define the dual field strength by

$$\tilde{F}_{\mu\nu}=\begin{cases} -\dfrac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} & \text{in Lorentzian signature}\ ,\\ \dfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} & \text{in Euclidean signature}\ ,\end{cases} \tag{2.3}$$

such that the self-dual and anti-self-dual parts of the field strength are given in both metric signatures by

$$F^{\pm}_{\mu\nu} = \frac{1}{2}\left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu}\right) \ . \tag{2.4}$$

3 $\mathcal{N} = 2$ gauged supergravity

We consider coupling an $\mathcal{N} = 2$ gravity multiplet to n_V vector multiplets. We use $\alpha = 1, \ldots, n_V$ to index the vector multiplets, as well as $I = 0, 1, \ldots, n_V$ to index all vector fields (including the graviphoton). The gravity multiplet is

$$\left(g_{\mu\nu}\ ,\ \psi^i_\mu\ ,\ A^0_\mu\right) \ , \tag{3.1}$$

where $g_{\mu\nu}$ is the metric, ψ^i_μ is an $SU(2)$ doublet of gravitinos, and A^0_μ is the graviphoton. The vector multiplets are given by

$$\left(A^\alpha_\mu\ ,\ \lambda^\alpha_i\ ,\ z^\alpha\right) \ , \tag{3.2}$$

where A^α_μ is a $U(1)$ vector field, λ^α_i is an $SU(2)$ doublet of gauginos, and z^α is a complex scalar. Note also that we consider Abelian vector multiplets, which means that the structure constants are ${f_{IJ}}^K = 0$. The field strengths are then simply given by $F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu$.

The interactions between the vector multiplets are specified by a prepotential $F(L)$, a holomorphic function of some complex scalars L^I with Weyl weight 2. These scalars arise in the off-shell superconformal construction of $\mathcal{N} = 2$ supergravity. We denote $M_I \equiv \partial_I F$ to be a derivative of the prepotential with respect to the holomorphic scalars. L^I and M_I can be presented together as a covariantly holomorphic section V:

$$V \equiv \begin{pmatrix} L^I \\ M_I \end{pmatrix} \ , \quad \nabla_{\bar{\alpha}} V \equiv \partial_{\bar{\alpha}} V - \frac{1}{2}(\partial_{\bar{\alpha}}\mathcal{K})V = 0 \ . \tag{3.3}$$

In gauge-fixing the superconformal symmetries to obtain a Poincaré supergravity theory, these scalars become subject to the constraint

$$\langle V, \bar{V}\rangle = L^I \bar{M}_I - M_I \bar{L}^I = i \ . \tag{3.4}$$

This constraint can be solved by defining:

$$\begin{pmatrix} L^I \\ M_I \end{pmatrix} = e^{\mathcal{K}/2} \begin{pmatrix} X^I \\ F_I \end{pmatrix} \ , \tag{3.5}$$

where X^I are complex scalars with Weyl weight 1 that are functions of the physical scalars z^α, and $F_I = \frac{\partial F(X)}{\partial X^I}$. The Kähler potential is then

$$\mathcal{K} = -\log\left[i\left(\bar{X}^I F_I - \bar{F}_I X^I\right)\right] \ . \tag{3.6}$$

The scalars X^I are not uniquely related to the physical scalars z^α. The physical scalars parameterize a special Kähler manifold of complex dimension n_V with Kähler potential given by (3.6), and we have freedom in how we choose the coordinates on this manifold. The corresponding Kähler metric is given by

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K} \ . \tag{3.7}$$

We also define the scalar kinetic mixing matrix $\mathcal{N}_{IJ}$ by

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + i \frac{N_{IK} X^K N_{JL} X^L}{N_{NM} X^N X^M} = \mathcal{R}_{IJ} + i \mathcal{I}_{IJ} \ , \tag{3.8}$$

where $N_{IJ} \equiv 2 \, \mathrm{Im} \, F_{IJ}$, and $\mathcal{R}, \mathcal{I}$ are the real and imaginary parts of $\mathcal{N}$, respectively.

The action for the bosonic fields is

$$S = \frac{1}{8\pi G_N} \int d^4x \sqrt{-g} \left[\frac{1}{2} R + \frac{1}{2} \mathrm{Im} \left(\mathcal{N}_{IJ} F^{+I}_{\mu\nu} F^{+J\mu\nu} \right) - g_{\alpha\bar{\beta}} \nabla^\mu z^\alpha \nabla_\mu \bar{z}^{\bar{\beta}} - g^2 \mathcal{P} \right] \ , \tag{3.9}$$

where the covariant derivative acting on the scalars is

$$\nabla_\mu z^\alpha = \partial_\mu z^\alpha + g A^I_\mu k^\alpha_I \ , \tag{3.10}$$

with k^α_I representing the Killing vectors of the special Kähler manifold, and g is the gauging parameter. (Note that it is distinct from the metric determinant, but hopefully context will make it clear which we mean.) Note that the vector kinetic term can be rewritten in several different ways:

$$\begin{aligned} e^{-1} \mathcal{L}_{\text{vector}} &\equiv \frac{1}{2} \mathrm{Im} \left(\mathcal{N}_{IJ} F^{+I}_{\mu\nu} F^{+J\mu\nu} \right) \\ &= \bar{\mathcal{N}}_{IJ} F^{-I}_{\mu\nu} F^{-J\mu\nu} - \mathcal{N}_{IJ} F^{+I}_{\mu\nu} F^{+J\mu\nu} \\ &= \frac{1}{4} \mathcal{I}_{IJ} F^I_{\mu\nu} F^{J\mu\nu} - \frac{i}{4} \mathcal{R}_{IJ} F^I_{\mu\nu} \tilde{F}^{J\mu\nu} \ . \end{aligned} \tag{3.11}$$

The scalar potential is

$$\mathcal{P} = g_{\alpha\bar{\beta}} k^\alpha_I k^{\bar{\beta}}_J \bar{L}^I L^J + \left(g^{\alpha\bar{\beta}} f^I_\alpha f^J_{\bar{\beta}} - 3 \bar{L}^I L^J \right) \vec{P}_I \cdot \vec{P}_J \ , \tag{3.12}$$

where $\vec{P}_I$ are the moment maps and f^I_α are the Kähler-covariant derivatives of the symplectic sections:

$$f^I_\alpha \equiv \nabla_\alpha L^I = \left(\partial_\alpha + \frac{1}{2} \partial_\alpha \mathcal{K} \right) L^I = e^{\mathcal{K}/2} \left(\partial_\alpha + \partial_\alpha \mathcal{K} \right) X^I = e^{\mathcal{K}/2} \nabla_\alpha X^I \ . \tag{3.13}$$

A useful rewriting of the potential is

$$\begin{aligned} \mathcal{P} &= g_{\alpha\bar{\beta}} k^\alpha_I k^{\bar{\beta}}_J \bar{L}^I L^J + \left(U^{IJ} - 3 \bar{L}^I L^J \right) \vec{P}_I \cdot \vec{P}_J \ , \\ U^{IJ} &= g^{\alpha\bar{\beta}} f^I_\alpha f^J_{\bar{\beta}} = g^{\alpha\bar{\beta}} \nabla_\alpha L^I \nabla_{\bar{\beta}} \bar{L}^J = -\frac{1}{2} \left(\mathcal{I}^{-1} \right)^{IJ} - \bar{L}^I L^J \ . \end{aligned} \tag{3.14}$$

We will now discuss the equations of motion for the bosonic fields in the Lagrangian. The (trace-reversed) Einstein equation is

$$R_{\mu\nu} = -\mathcal{I}_{IJ} \left(F^I_{\mu\rho} F^J{}_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F^I_{\rho\sigma} F^{J\rho\sigma} \right) + 2 g_{\alpha\bar{\beta}} \nabla_\mu z^\alpha \nabla_\nu \bar{z}^{\bar{\beta}} + g_{\mu\nu} g^2 \mathcal{P} \ . \tag{3.15}$$

We are eventually going to be interested in models with no gauging of the special Kähler isometries, such as the STU model, in which case we find $k_I^\alpha = 0$. The Maxwell equation is then

$$\nabla_\mu \left(\mathcal{I}_{IJ} F^{I\mu\nu} - i\mathcal{R}_{IJ} \tilde{F}^{I\mu\nu} \right) = 0 \ , \tag{3.16}$$

while the Bianchi identity is simply $\nabla_\mu \tilde{F}^{I\mu\nu} = 0$. Finally, the scalar equation of motion is

$$\nabla^\mu \left(g_{\alpha\bar{\beta}} \nabla_\mu \bar{z}^{\bar{\beta}} \right) = -\frac{1}{4} \partial_\alpha \mathcal{I}_{IJ} F^I_{\mu\nu} F^{J\mu\nu} + \frac{i}{4} \partial_\alpha \mathcal{R}_{IJ} F^I_{\mu\nu} \tilde{F}^{J\mu\nu} + \partial_\alpha g_{\delta\bar{\beta}} \nabla^\mu z^\delta \nabla_\mu \bar{z}^{\bar{\beta}} + g^2 \partial_\alpha \mathcal{P} \ . \tag{3.17}$$

In the case where $k_I^\alpha = 0$, the supersymmetry variations of the fermionic fields are as follows:

$$\begin{aligned} \delta\psi^i_\mu &= \mathcal{D}_\mu \epsilon^i + g P_I^{ij} L^I \gamma_\mu \epsilon_j + \frac{1}{4} \mathcal{I}_{IJ} L^I F^{-J}_{ab} \gamma^{ab} \gamma_\mu \varepsilon^{ij} \epsilon_j \ , \\ \mathcal{D}_\mu \epsilon^i &= \left(\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} + \frac{i}{2} \mathcal{A}_\mu \right) \epsilon^i + g A^I_\mu {P_{Ij}}^i \epsilon^j \ , \\ \delta\lambda^\alpha_i &= -\frac{1}{2} g^{\alpha\bar{\beta}} f^I_{\bar{\beta}} \mathcal{I}_{IJ} F^{-J}_{ab} \gamma^{ab} \varepsilon_{ij} \epsilon^j + \gamma^\mu \nabla_\mu z^\alpha \epsilon_i - 2g P_{Iij} g^{\alpha\bar{\beta}} f^I_{\bar{\beta}} \epsilon^j \ , \end{aligned} \tag{3.18}$$

where we have defined the Kähler connection $\mathcal{A}_\mu$ by

$$\mathcal{A}_\mu = -\frac{i}{2} (\partial_\alpha \mathcal{K} \partial_\mu z^\alpha - \partial_{\bar{\alpha}} \mathcal{K} \partial_\mu \bar{z}^{\bar{\alpha}}) \ , \tag{3.19}$$

and we have also defined

$${P_{Ii}}^j = \frac{i}{2} \vec{P}_I \cdot {\vec{\sigma}_i}^{\ j} \ , \tag{3.20}$$

where the Pauli matrices ${(\sigma_{1,2,3})_i}^j$ are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{3.21}$$

Note that the $SU(2)$ indices on these matrices are raised on the left and lowered on the right, such that

$$\vec{\sigma}_{ij} \equiv {\vec{\sigma}_i}^{\ k} \varepsilon_{kj} \ , \quad \vec{\sigma}^{ij} \equiv \varepsilon^{ik} {\vec{\sigma}_k}^{\ j} \ . \tag{3.22}$$

The supersymmetry parameters ϵ^i form an $SU(2)$ doublet of left-handed Weyl spinors, while their Majorana conjugates ϵ_i form an $SU(2)$ doublet of right-handed Weyl spinors:

$$\begin{aligned} \gamma_5 \epsilon^i &= \epsilon^i \ , \\ \gamma_5 \epsilon_i &= -\epsilon_i \ . \end{aligned} \tag{3.23}$$

We can go from one to the other via the Majorana conjugation operation:

$$\epsilon_i = \gamma_0 \mathcal{C}^{-1} (\epsilon^i)^* \ , \tag{3.24}$$

where $\mathcal{C}$ is the charge conjugation matrix, whose definition depends on your explicit gamma matrix conventions. In terms of these Majorana conjugates, the supersymmetry variations are:

$$\begin{aligned}
\delta\psi_{i\mu} &= \mathcal{D}_\mu\epsilon_i + gP_{Iij}\bar{L}^I\gamma_\mu\epsilon^j + \frac{1}{4}\mathcal{I}_{IJ}\bar{L}^I F^{+J}_{ab}\gamma^{ab}\gamma_\mu\varepsilon_{ij}\epsilon^j \ , \\
\mathcal{D}_\mu\epsilon_i &= \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab} - \frac{i}{2}\mathcal{A}_\mu\right)\epsilon_i - gA^I_\mu P_{Ii}{}^j\epsilon_j \ , \\
\delta\lambda^{i\bar{\alpha}} &= -\frac{1}{2}g^{\bar{\alpha}\beta}f^I_\beta\mathcal{I}_{IJ}F^{+J}_{ab}\gamma^{ab}\varepsilon^{ij}\epsilon_j + \gamma^\mu\nabla_\mu\bar{z}^{\bar{\alpha}}\epsilon^i - 2gP^{ij}_I g^{\bar{\alpha}\beta}f^I_\beta\epsilon_j \ .
\end{aligned} \tag{3.25}$$

In Lorentzian supergravity it doesn't matter if we find supersymmetric solutions using the variations (3.18) or (3.25); demanding that one set vanishes immediately implies the other vanishes as well, since they are Majorana conjugates. Crucially, this will no longer be the case in Euclidean signature.

4 Euclideanization

Our method of Euclideanization will follow the one proposed in [1]. That is, we will analytically continue our Lorentzian supergravity theory into a Euclidean one by means of a Wick-rotation of the tangent space time direction:

$$x^0 \to -ix^4 \ . \tag{4.1}$$

Correspondingly, the time component of all vector fields must also be Wick-rotated. In particular,

$$A^I_0 \to iA^I_4 \ , \quad \gamma_0 \to i\gamma_4 \ , \tag{4.2}$$

and so on. We must also take care to Wick-rotate the action accordingly, due to the integration over the time coordinate in the measure.

A subtlety in this procedure is that the fermionic fields ψ^i_μ, λ^i_α, and ϵ^i become independent from their would-be Majorana conjugates $\psi_{i\mu}$, $\lambda^{i\bar{\alpha}}$, and ϵ_i. In Lorentzian spacetimes, fermions and their Majorana conjugates fall in conjugate representations of the four-dimensional Lorentzian isometry group $SO(3,1)$. For example, a spinor field with spin-$\frac{1}{2}$ can fall into either the $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$ representations of the Lorentz group, and these two representations are in fact conjugate representations, in the sense that the action of the charge conjugation operator takes us from one to the other. However, the Euclidean isometry group is $SO(4)\cong SU(2)\times SU(2)$, which does not admit pairs of conjugate representations for the spinor fields since the group elements of each $SU(2)$ act independently of one another.

As a result, we must treat all fermions and their would-be conjugates in Lorentzian signature as truly independent fields. Furthermore, since fermions and bosons transform into one another via the action of the supersymmetry generators, we must also allow the scalars z^α and $\bar{z}^\alpha$ to be *independent* complex scalars, as well as allowing for the gauge fields A^I_μ to be complex. Using the notation of [1], we will denote the conjugate scalars now by $\tilde{z}^\alpha$ instead of $\bar{z}^\alpha$ to emphasize their independence.

Once we account for all of the Euclideanization ingredients described above, we find that the action for the bosonic fields in our Euclidean theory is given by

$$S = \frac{1}{8\pi G_N} \int d^4x \sqrt{g} \left[-\frac{1}{2}R - \frac{1}{4}\mathcal{I}_{IJ}F^I_{\mu\nu}F^{J\mu\nu} + \frac{i}{4}\mathcal{R}_{IJ}F^I_{\mu\nu}\tilde{F}^{J\mu\nu} + g_{\alpha\bar{\beta}}\nabla^\mu z^\alpha \nabla_\mu \tilde{z}^{\bar{\beta}} + g^2\mathcal{P} \right] , \tag{4.3}$$

where we take the explicit forms of the Kähler potential $\mathcal{K}$ from (3.6), the Kähler metric $g_{\alpha\bar{\beta}}$ from (3.7), the matrices $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ from (3.8), and the potential $\mathcal{P}$ from (3.12), and we replace all instances of $\bar{z}^\alpha$ with $\tilde{z}^\alpha$. The supersymmetry variations of the fermionic fields are as follows:

$$\begin{aligned} \delta\psi^i_\mu &= \mathcal{D}_\mu\epsilon^i + gP^{ij}_I L^I \gamma_\mu \epsilon_j + \frac{1}{4}\mathcal{I}_{IJ}L^I F^{-J}_{ab}\gamma^{ab}\gamma_\mu \varepsilon^{ij}\epsilon_j , \\ \delta\psi_{i\mu} &= \mathcal{D}_\mu\epsilon_i + gP_{Iij}\bar{L}^I\gamma_\mu\epsilon^j + \frac{1}{4}\mathcal{I}_{IJ}\bar{L}^I(F^{+J}_{ab})^*\gamma^{ab}\gamma_\mu\varepsilon_{ij}\epsilon^j , \\ \delta\lambda^\alpha_i &= -\frac{1}{2}g^{\alpha\bar{\beta}}f^I_{\bar{\beta}}\mathcal{I}_{IJ}F^{-J}_{ab}\gamma^{ab}\varepsilon_{ij}\epsilon^j + \gamma^\mu\nabla_\mu z^\alpha\epsilon_i - 2gP_{Iij}g^{\alpha\bar{\beta}}f^I_{\bar{\beta}}\epsilon^j , \\ \delta\lambda^{i\bar{\alpha}} &= -\frac{1}{2}g^{\bar{\alpha}\beta}f^I_\beta\mathcal{I}_{IJ}(F^{+J}_{ab})^*\gamma^{ab}\varepsilon^{ij}\epsilon_j + \gamma^\mu\nabla_\mu\tilde{z}^{\bar{\alpha}}\epsilon^i - 2gP^{ij}_I g^{\bar{\alpha}\beta}f^I_\beta\epsilon_j , \end{aligned} \tag{4.4}$$

where the supercovariant derivatives act via

$$\begin{aligned} \mathcal{D}_\mu\epsilon^i &= \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab} + \frac{i}{2}\mathcal{A}_\mu\right)\epsilon^i + gA^I_\mu P_{Ij}{}^i\epsilon^j , \\ \mathcal{D}_\mu\epsilon_i &= \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab} - \frac{i}{2}\mathcal{A}_\mu\right)\epsilon_i - g(A^I_\mu)^* P_{Ii}{}^j\epsilon_j , \end{aligned} \tag{4.5}$$

and the Kähler connection in Euclidean signature is now given by

$$\mathcal{A}_\mu = -\frac{i}{2}\left(\frac{\partial\mathcal{K}}{\partial z^\alpha}\partial_\mu z^\alpha - \frac{\partial\mathcal{K}}{\partial\tilde{z}^\alpha}\partial_\mu\tilde{z}^\alpha\right) . \tag{4.6}$$

Note that we now have to take complex conjugates in the conjugate variations where appropriate due to the complexification of the gauge fields and the independence of z^α and $\tilde{z}^\alpha$.

We should note that a lot of the existing studies of Euclidean supergravity [1, 2, 5–7] don't explicitly write out the complex conjugation on the gauge fields A^I_μ and instead just keep them implicit or consider ansatzes in which the gauge fields are real. Additionally, there should in principle also be complex conjugations of the vielbein fields where appropriate, but in the interest of only considering real metrics we will forego this extra level of rigor.

5 (Euclidean) STU model

We work with $n_V = 3$ vector multiplets. The complex scalars z^α in these vector multiplets parameterize the Kähler manifold $\mathcal{M}_V$, which for the STU model is three copies of the Poincaré disk:

$$\mathcal{M}_V = \left[\frac{SU(1,1)}{U(1)}\right]^3 . \tag{5.1}$$

Accordingly, the scalars must take values on the Poincaré disk and thus they satisfy the condition $|z^\alpha| < 1$. The projective scalars X^I are related to the physical scalars z^α by

$$\begin{aligned} X^0 &= \frac{1}{2\sqrt{2}}(1-z^1)(1-z^2)(1-z^3)\ , \qquad & X^1 &= \frac{1}{2\sqrt{2}}(1-z^1)(1+z^2)(1+z^3)\ , \\ X^2 &= \frac{1}{2\sqrt{2}}(1+z^1)(1-z^2)(1+z^3)\ , \qquad & X^3 &= \frac{1}{2\sqrt{2}}(1+z^1)(1+z^2)(1-z^3)\ . \end{aligned} \tag{5.2}$$

The prepotential F for the STU model is given by

$$F = -2i\sqrt{X^0X^1X^2X^3} = -\frac{i}{4}\left(1-(z^1)^2\right)\left(1-(z^2)^2\right)\left(1-(z^3)^2\right)\ . \tag{5.3}$$

The Kähler potential $\mathcal{K}$ and the Kähler metric $g_{\alpha\bar{\beta}}$ are correspondingly given by

$$\mathcal{K} = -\sum_{\alpha=1}^{3}\log\left[(1-|z^\alpha|^2)\right]\ , \qquad g_{\alpha\bar{\beta}} = \frac{\delta_{\alpha\bar{\beta}}}{(1-|z^\alpha|^2)^2}\ . \tag{5.4}$$

We also define the superpotential

$$\mathcal{V} = 2\left(z^1z^2z^3 - 1\right)\ . \tag{5.5}$$

The scalar potential can be expressed in terms of this as

$$\mathcal{P} = \frac{1}{2}e^{\mathcal{K}}\left(g^{\alpha\bar{\beta}}\nabla_\alpha\mathcal{V}\nabla_{\bar{\beta}}\bar{\mathcal{V}} - 3\mathcal{V}\bar{\mathcal{V}}\right) = 2\left(3 - \sum_{\alpha=1}^{3}\frac{2}{1-|z^\alpha|^2}\right)\ . \tag{5.6}$$

Finally, the moment maps are

$$P_I^1 = P_I^2 = 0\ , \quad P_I^3 = -1\ , \tag{5.7}$$

for any value of I.

When we Euclideanize, everything said above remains the same except we now have to replace $\bar{z}^\alpha$ with $\tilde{z}^\alpha$. Both sets of scalars z^α and $\tilde{z}^\alpha$ now parameterize two different copies of the Poincaré disk and so they must satisfy $|z^\alpha| < 1$ and $|\tilde{z}^\alpha| < 1$. The supersymmetry variations in the Euclideanized STU model are as follows:

$$\begin{aligned} \delta\psi^i_\mu &= \mathcal{D}_\mu\epsilon^i + \frac{i}{2}gP_I^3L^I\gamma_\mu(\sigma_3)^{ij}\epsilon_j + \frac{1}{4}\mathcal{I}_{IJ}L^IF^{-J}_{ab}\gamma^{ab}\gamma_\mu\varepsilon^{ij}\epsilon_j\ , \\ \delta\psi_{i\mu} &= \mathcal{D}_\mu\epsilon_i + \frac{i}{2}gP_I^3\bar{L}^I\gamma_\mu(\sigma_3)_{ij}\epsilon^j + \frac{1}{4}\mathcal{I}_{IJ}\bar{L}^I(F^{+J}_{ab})^*\gamma^{ab}\gamma_\mu\varepsilon_{ij}\epsilon^j\ , \\ \delta\lambda^\alpha_i &= -\frac{1}{2}g^{\alpha\bar{\beta}}f^I_{\bar{\beta}}\mathcal{I}_{IJ}F^{-J}_{ab}\gamma^{ab}\varepsilon_{ij}\epsilon^j + \gamma^\mu\nabla_\mu z^\alpha\epsilon_i - igg^{\alpha\bar{\beta}}f^I_{\bar{\beta}}P^3_I(\sigma_3)_{ij}\epsilon^j\ , \\ \delta\lambda^{i\bar{\alpha}} &= -\frac{1}{2}g^{\bar{\alpha}\beta}f^I_\beta\mathcal{I}_{IJ}(F^{+J}_{ab})^*\gamma^{ab}\varepsilon^{ij}\epsilon_j + \gamma^\mu\nabla_\mu\tilde{z}^{\bar{\alpha}}\epsilon^i - igg^{\bar{\alpha}\beta}f^I_\beta P^3_I(\sigma_3)^{ij}\epsilon_j\ , \end{aligned} \tag{5.8}$$

where the covariant derivatives are given by

$$\begin{aligned} \mathcal{D}_\mu\epsilon^i &= \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab} + \frac{i}{2}\mathcal{A}_\mu\right)\epsilon^i + \frac{i}{2}gP^3_IA^I_\mu(\sigma_3)_j{}^i\epsilon^j\ , \\ \mathcal{D}_\mu\epsilon_i &= \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab} - \frac{i}{2}\mathcal{A}_\mu\right)\epsilon_i - \frac{i}{2}gP^3_I(A^I_\mu)^*(\sigma_3)_i{}^j\epsilon_j\ . \end{aligned} \tag{5.9}$$

With these variations, one can show that the STU model admits a maximally supersymmetric vacuum AdS_4 solution wherein all scalars and gauge fields are turned off. The size L of this AdS_4 vacuum is related to the gauging parameter g by

$$g = \frac{1}{\sqrt{2}L} \ . \tag{5.10}$$

6 STU model with S^3 boundary

In this section, we reproduce the results of [1] by constructing Euclidean BPS solutions in the STU model whose boundary is that of a round S^3. We derive the BPS conditions and perform holographic renormalization to compute the on-shell action for such BPS solutions, and we establish a match with the partition function of the dual ABJM theory.

6.1 Ansatz and variations

The ansatz is

$$\begin{aligned} ds^2 &= \frac{1}{4}L^2 e^{2A(r)}\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)+e^{2B(r)}dr^2 \ , \\ A^I &= 0 \ , \\ z^\alpha &= z^\alpha(r) \ , \\ \tilde{z}^\alpha &= \tilde{z}^\alpha(r) \ , \end{aligned} \tag{6.1}$$

such that all gauge fields are turned off and all scalars are functions only of the radial coordinate r. The functions A and B are both functions only of r as well. The quantities σ_i, through an abuse of notation, are not Pauli matrices; they are Maurer-Cartan one-forms. Specifically, they are the left-invariant one-forms of $SO(3)$, given by

$$\begin{aligned} \sigma_1 &= \sin\theta\cos\psi\, d\phi - \sin\psi\, d\theta \ , \\ \sigma_2 &= \sin\theta\sin\psi\, d\phi + \cos\psi\, d\theta \ , \\ \sigma_3 &= \cos\theta\, d\phi + d\psi \ , \end{aligned} \tag{6.2}$$

where the coordinates (ϕ,θ,ψ) have periodicities $(2\pi,\pi,4\pi)$, respectively. These one-forms satisfy the relationship

$$d\sigma_a = \frac{1}{2}\varepsilon_{abc}\sigma_b\wedge\sigma_c \ . \tag{6.3}$$

Using the explicit form of the one-forms, one can show that they square in quadrature to the metric of a unit S^3, albeit written in a slightly unconventional form:

$$ds^2_{S^3} = \frac{1}{4}(\sigma_1^2+\sigma_2^2+\sigma_3^2) = \frac{1}{4}\left(d\phi^2+d\theta^2+d\psi^2+2\cos\theta\, d\phi\, d\psi\right) \ , \tag{6.4}$$

and so our metric ansatz above is just a complicated way of saying that we want our spacetime to have an asymptotic S^3 boundary. The reason we are going through all of this is to make the squashed sphere (which we'll do later) easier.

The vielbein are

$$e^i = \frac{1}{2}Le^A\sigma_i \ , \quad e^4 = e^B dr \ , \tag{6.5}$$

and the spin connections are

$$\omega^{ij} = \frac{1}{2}\varepsilon^{ijk}\sigma_k \ , \quad \omega^{i4} = A' e^{-B} e^i = \frac{1}{2} L e^{A-B} A' \sigma_i \ . \tag{6.6}$$

If we keep the gauge fields turned off, the variations are given by

$$\begin{aligned}
\delta\psi^i_\mu &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} + \frac{i}{2}\mathcal{A}_\mu\right)\epsilon^i + \frac{i}{2\sqrt{2}L}P^3_I L^I \gamma_\mu (\sigma_3)^{ij}\epsilon_j \ , \\
\delta\psi_{i\mu} &= \left(\partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab} - \frac{i}{2}\mathcal{A}_\mu\right)\epsilon^i + \frac{i}{2\sqrt{2}L}P^3_I \bar{L}^I \gamma_\mu (\sigma_3)_{ij}\epsilon^j \ , \\
\delta\lambda^\alpha_i &= e^{-B}\partial_r z^\alpha \gamma_4 \epsilon_i - \frac{i}{\sqrt{2}L} g^{\alpha\bar{\beta}} f^I_{\bar{\beta}} P^3_I (\sigma_3)_{ij}\epsilon^j \ , \\
\delta\lambda^{i\bar{\alpha}} &= e^{-B}\partial_r \bar{z}^{\bar{\alpha}} \gamma_4 \epsilon^i - \frac{i}{\sqrt{2}L} g^{\bar{\alpha}\beta} f^I_\beta P^3_I (\sigma_3)^{ij}\epsilon_j \ .
\end{aligned} \tag{6.7}$$

The $\mu = r$ component of the gravitino variations can be written explicitly as

$$\begin{aligned}
\delta\psi^i_r &= \left(\partial_r + \frac{i}{2}\mathcal{A}_r\right)\epsilon^i + \frac{i}{2\sqrt{2}L} e^B P^3_I L^I (\sigma_3)^{ij}\gamma_4\epsilon_j \ , \\
\delta\psi_{ri} &= \left(\partial_r - \frac{i}{2}\mathcal{A}_r\right)\epsilon_i + \frac{i}{2\sqrt{2}L} e^B P^3_I \bar{L}^I (\sigma_3)_{ji}\gamma_4\epsilon^j \ .
\end{aligned} \tag{6.8}$$

The components along the S^3 are easier to understand if we contract them with the corresponding dual Killing vector fields. That is, we define

$$X_{\sigma_i} \equiv (K_i)^\mu X_\mu \ , \tag{6.9}$$

where K_i is the dual Killing vector associated with each Mauren-Cartan form σ_i. When we say they are dual, we mean it in the sense that $(K_i)^\mu (\sigma_j)_\mu = \delta_{ij}$. Explicitly, these are given by

$$\begin{aligned}
K_1 &= \frac{\cos\psi}{\sin\theta}\,\partial_\phi - \sin\psi\,\partial_\theta - \cot\theta\cos\psi\,\partial_\psi \ , \\
K_2 &= \frac{\sin\psi}{\sin\theta}\,\partial_\phi + \cos\psi\,\partial_\theta - \cot\theta\sin\psi\,\partial_\psi \ , \\
K_3 &= \partial_\psi \ .
\end{aligned} \tag{6.10}$$

Contracting these with the spin connection yields the identity

$$\omega_{\sigma_a}{}^{bc} = \frac{1}{2}\varepsilon^{abc} \ . \tag{6.11}$$

With this in mind, we write out the σ_1 component of the gravitino variations explicitly:

$$\begin{aligned}
\delta\psi^i_{\sigma_1} &= \left(\partial_{\sigma_1} + \frac{1}{4}\gamma_{23} + \frac{1}{4}Le^{A-B}A'\gamma_{14}\right)\epsilon^i + \frac{i}{4\sqrt{2}} e^A P^3_I L^I \gamma_1 (\sigma_3)^{ij}\epsilon_j \ , \\
\delta\psi_{\sigma_1 i} &= \left(\partial_{\sigma_1} + \frac{1}{4}\gamma_{23} + \frac{1}{4}Le^{A-B}A'\gamma_{14}\right)\epsilon_i + \frac{i}{4\sqrt{2}} e^A P^3_I \bar{L}^I \gamma_1 (\sigma_3)_{ji}\epsilon^j \ .
\end{aligned} \tag{6.12}$$

The others are obtained easily by cyclic permutations. We can actually make this easier to permute by rewriting it as

$$\begin{aligned}
\delta\psi^i_{\sigma_1} &= \left(\partial_{\sigma_1} - \frac{1}{4}\gamma_{14} + \frac{1}{4}Le^{A-B}A'\gamma_{14}\right)\epsilon^i + \frac{i}{4\sqrt{2}} e^A P^3_I L^I \gamma_1 (\sigma_3)^{ij}\epsilon_j \ , \\
\delta\psi_{\sigma_1 i} &= \left(\partial_{\sigma_1} + \frac{1}{4}\gamma_{14} + \frac{1}{4}Le^{A-B}A'\gamma_{14}\right)\epsilon_i + \frac{i}{4\sqrt{2}} e^A P^3_I \bar{L}^I \gamma_1 (\sigma_3)_{ji}\epsilon^j \ .
\end{aligned} \tag{6.13}$$

6.2 Projectors and BPS conditions

We want to look for $\frac{1}{2}$-BPS solutions. All of the variations in the previous section appear with γ_4 and σ_3 in front of the SUSY parameters, and based on the structure of these we will guess projectors of the form

$$\begin{aligned} \gamma_4 \epsilon_i &= iM(r)(\sigma_3)_{ij}\epsilon^j \ , \\ \gamma_4 \epsilon^i &= i\widetilde{M}(r)(\sigma_3)^{ij}\epsilon_j \ , \end{aligned} \tag{6.14}$$

where $M(r)$ and $\widetilde{M}(r)$ are some functions that satisfy $M\widetilde{M} = 1$. Note that a useful identity in what follows is $(\sigma_3)^{ij}(\sigma_3)_{jk} = -\delta^i_k$. With this, the gaugino variations become

$$\begin{aligned} \delta\lambda^\alpha_i &= i\left(Me^{-B}\partial_r z^\alpha - \frac{1}{\sqrt{2}L} g^{\alpha\bar\beta} f^I_{\bar\beta} P^3_I\right)(\sigma_3)_{ij}\epsilon^j \ , \\ \delta\lambda^{i\bar\alpha} &= i\left(\widetilde{M}e^{-B}\partial_r \bar z^{\bar\alpha} - \frac{1}{\sqrt{2}L} g^{\bar\alpha\beta} f^I_{\beta} P^3_I\right)(\sigma_3)^{ji}\epsilon_j \ . \end{aligned} \tag{6.15}$$

The radial component of the gravitino variations become

$$\begin{aligned} \delta\psi^i_r &= \left(\partial_r + \frac{i}{2}\mathcal{A}_r + \frac{1}{2\sqrt{2}L} e^B P^3_I L^I\right)\epsilon^i \ , \\ \delta\psi_{ri} &= \left(\partial_r - \frac{i}{2}\mathcal{A}_r + \frac{1}{2\sqrt{2}L} e^B P^3_I \bar L^I\right)\epsilon_i \ . \end{aligned} \tag{6.16}$$

Lastly, the σ_1 components of the gravitino variations are

$$\begin{aligned} \delta\psi^i_{\sigma_1} &= \partial_{\sigma_1}\epsilon^i + \frac{i}{4}\left(-\widetilde{M} + \widetilde{M}Le^{A-B}A' + \frac{1}{\sqrt{2}} e^A P^3_I L^I\right)\gamma_1(\sigma_3)^{ij}\epsilon_j \ , \\ \delta\psi_{\sigma_1 i} &= \partial_{\sigma_1}\epsilon_i + \frac{i}{4}\left(M + MLe^{A-B}A' + \frac{1}{\sqrt{2}} e^A P^3_I \bar L^I\right)\gamma_1(\sigma_3)_{ij}\epsilon^j \ . \end{aligned} \tag{6.17}$$

Similar expressions hold for the σ_2 and σ_3 components. Setting all variations to zero, we arrive at the BPS conditions we want. The gaugino variations yield the following conditions:

$$\begin{aligned} 0 &= Me^{-B}\partial_r z^\alpha - \frac{1}{\sqrt{2}L} g^{\alpha\bar\beta} f^I_{\bar\beta} P^3_I \ , \\ 0 &= \widetilde{M}e^{-B}\partial_r \bar z^{\bar\alpha} - \frac{1}{\sqrt{2}L} g^{\bar\alpha\beta} f^I_{\beta} P^3_I \ . \end{aligned} \tag{6.18}$$

The radial components of the gravitino variations tell us the radial dependence of the spinors:

$$\begin{aligned} 0 &= \left(\partial_r + \frac{i}{2}\mathcal{A}_r + \frac{1}{2\sqrt{2}L} e^B P^3_I L^I\right)\epsilon^i \ , \\ 0 &= \left(\partial_r - \frac{i}{2}\mathcal{A}_r + \frac{1}{2\sqrt{2}L} e^B P^3_I \bar L^I\right)\epsilon_i \ . \end{aligned} \tag{6.19}$$

Finally, the components of the gravitino variations along the S^3 tell us that, for any value of $a = 1, 2, 3$, we have

$$\begin{aligned} 0 &= \partial_{\sigma_a}\epsilon^i + \frac{i}{4}\left(-\widetilde{M} + \widetilde{M}Le^{A-B}A' + \frac{1}{\sqrt{2}} e^A P^3_I L^I\right)\gamma_a(\sigma_3)^{ij}\epsilon_j \ , \\ 0 &= \partial_{\sigma_a}\epsilon_i + \frac{i}{4}\left(M + MLe^{A-B}A' + \frac{1}{\sqrt{2}} e^A P^3_I \bar L^I\right)\gamma_a(\sigma_3)_{ij}\epsilon^j \ . \end{aligned} \tag{6.20}$$

There are two ways we can proceed. We can either be agnostic about the conformal Killing spinors on S^3 and use integrability conditions, or we can be explicit about how the spinors are linear combinations of S^3 conformal Killing spinors. Let's try the former. First, we define

$$\begin{aligned}\phi &\equiv M + MLe^{A-B}A' + \frac{1}{\sqrt{2}}e^{A}P_I^3\bar{L}^I \ ,\\ \tilde{\phi} &\equiv -\widetilde{M} + \widetilde{M}Le^{A-B}A' + \frac{1}{\sqrt{2}}e^{A}P_I^3L^I \ .\end{aligned} \tag{6.21}$$

Then, the S^3 conditions are simply

$$\partial_{\sigma_a}\epsilon^i = -\frac{i}{4}\tilde{\phi}\gamma_a(\sigma_3)^{ij}\epsilon_j \ , \quad \partial_{\sigma_a}\epsilon_i = -\frac{i}{4}\phi\gamma_a(\sigma_3)_{ij}\epsilon^j \ . \tag{6.22}$$

The commutator of derivatives acts on the spinors by

$$[\partial_{\sigma_a}, \partial_{\sigma_b}]\epsilon^i = -\frac{1}{8}\phi\tilde{\phi}\gamma_{ab}\epsilon^i \ , \quad [\partial_{\sigma_a}, \partial_{\sigma_b}]\epsilon_i = -\frac{1}{8}\phi\tilde{\phi}\gamma_{ab}\epsilon_i \ . \tag{6.23}$$

The derivatives along the Mauren-Cartan forms satisfy the Jacobi identity

$$[\partial_{\sigma_a}, \partial_{\sigma_b}] = -\varepsilon^{abc}\partial_{\sigma_c} \ . \tag{6.24}$$

This means that we must set

$$[\partial_{\sigma_a}, \partial_{\sigma_b}]\epsilon^i = -\varepsilon^{abc}\partial_{\sigma_c}\epsilon^i \ , \quad [\partial_{\sigma_a}, \partial_{\sigma_b}]\epsilon_i = -\varepsilon^{abc}\partial_{\sigma_c}\epsilon_i \ , \tag{6.25}$$

which tells us the following two integrability conditions:

$$\begin{aligned}\phi\tilde{\phi}\gamma_{ab}\epsilon^i &= -2i\tilde{\phi}\varepsilon^{abc}\gamma_c(\sigma_3)^{ij}\epsilon_j \ ,\\ \phi\tilde{\phi}\gamma_{ab}\epsilon_i &= -2i\phi\varepsilon^{abc}\gamma_c(\sigma_3)_{ij}\epsilon^j \ .\end{aligned} \tag{6.26}$$

Then, we use the fact that $\gamma_{ab} = -\varepsilon^{abc}\gamma_{c4}\gamma_5$ along with the spinor chiralities and the projectors to write these integrability conditions as

$$\begin{aligned}-i\widetilde{M}\phi\tilde{\phi}\varepsilon^{abc}\gamma_c(\sigma_3)^{ij}\epsilon_j &= -2i\tilde{\phi}\varepsilon^{abc}\gamma_c(\sigma_3)^{ij}\epsilon_j \ ,\\ iM\phi\tilde{\phi}\varepsilon^{abc}\gamma_c(\sigma_3)_{ij}\epsilon^j &= -2i\phi\varepsilon^{abc}\gamma_c(\sigma_3)_{ij}\epsilon^j \ .\end{aligned} \tag{6.27}$$

This is true for any values of $a, b = 1, 2, 3$, and so we find that

$$\tilde{\phi}\left(\widetilde{M}\phi - 2\right) = 0 \ , \quad \phi\left(M\tilde{\phi} + 2\right) = 0 \ . \tag{6.28}$$

There are two possible solutions:

$$\phi = \tilde{\phi} = 0 \ , \quad \text{or} \quad \phi - 2M = \tilde{\phi} + 2\widetilde{M} = 0 \ . \tag{6.29}$$

If we write these out explicitly, they take the form

$$\begin{aligned}0 &= \xi M + MLe^{A-B}A' + \frac{1}{\sqrt{2}}e^{A}P_I^3\bar{L}^I \ ,\\ 0 &= -\xi\widetilde{M} + \widetilde{M}Le^{A-B}A' + \frac{1}{\sqrt{2}}e^{A}P_I^3L^I \ ,\end{aligned} \tag{6.30}$$

where $\xi = \pm 1$ picks out which possible solution we have.

6.3 Solutions

To summarize the above, if we take equations (6.18) and (6.30), we have the following Euclidean BPS conditions:

$$\begin{aligned}
0 &= Me^{-B}\partial_r z^\alpha - \frac{1}{\sqrt{2}L} g^{\alpha\bar{\beta}} f^I_{\bar{\beta}} P^3_I \ , \\
0 &= \widetilde{M}e^{-B}\partial_r \tilde{z}^{\bar{\alpha}} - \frac{1}{\sqrt{2}L} g^{\bar{\alpha}\beta} f^I_{\beta} P^3_I \ , \\
0 &= \xi M + MLe^{A-B}A' + \frac{1}{\sqrt{2}} e^A P^3_I \bar{L}^I \ , \\
0 &= -\xi \widetilde{M} + \widetilde{M}Le^{A-B}A' + \frac{1}{\sqrt{2}} e^A P^3_I L^I \ ,
\end{aligned} \tag{6.31}$$

where $\xi = \pm 1$ and $M\widetilde{M} = 1$. We can solve the third and fourth equations (i.e. the gravitino variations) for the projectors explicitly:

$$M = -\frac{\frac{1}{\sqrt{2}} e^A P^3_I \bar{L}^I}{\xi + Le^{A-B}A'} \ , \quad \widetilde{M} = -\frac{\frac{1}{\sqrt{2}} e^A P^3_I L^I}{-\xi + Le^{A-B}A'} \ . \tag{6.32}$$

By demanding that $M\widetilde{M} = 1$, we find the following projector-independent condition:

$$0 = 1 - L^2(A')^2 e^{2A-2B} + \frac{1}{2} e^{2A} (P^3_I L^I)(P^3_J \bar{L}^J) \ . \tag{6.33}$$

If we now eliminate the projectors from the first two equations (i.e. the gaugino variations), we find that

$$\begin{aligned}
Le^{A-B} P^3_I \bar{L}^I \partial_r z^\alpha &= (-\xi - Le^{A-B}A') g^{\alpha\bar{\beta}} f^I_{\bar{\beta}} P^3_I \ , \\
Le^{A-B} P^3_I L^I \partial_r \tilde{z}^{\bar{\alpha}} &= (\xi - Le^{A-B}A') g^{\bar{\alpha}\beta} f^I_{\beta} P^3_I \ .
\end{aligned} \tag{6.34}$$

If we take all of these conditions and write out the symplectic sections explicitly in terms of the scalars, we finally end up with the following set of conditions:

$$\begin{aligned}
Le^{A-B}(1 - \tilde{z}^1\tilde{z}^2\tilde{z}^3)\partial_r z^\alpha &= (-\xi - Le^{A-B}A')(1 - z^\alpha \tilde{z}^\alpha)\left(z^\alpha - \frac{\tilde{z}^1\tilde{z}^2\tilde{z}^3}{\tilde{z}^\alpha}\right) \ , \\
Le^{A-B}(1 - z^1 z^2 z^3)\partial_r \tilde{z}^\alpha &= (\xi - Le^{A-B}A')(1 - z^\alpha \tilde{z}^\alpha)\left(\tilde{z}^\alpha - \frac{z^1 z^2 z^3}{z^\alpha}\right) \ , \\
-1 &= -L^2(A')^2 e^{2A-2B} + e^{2A} \frac{(1 - z^1 z^2 z^3)(1 - \tilde{z}^1\tilde{z}^2\tilde{z}^3)}{(1 - z^1\tilde{z}^1)(1 - z^2\tilde{z}^2)(1 - z^3\tilde{z}^3)} \ .
\end{aligned} \tag{6.35}$$

This is the final set of BPS conditions. Under the following identification:

$$z^\alpha_{\rm FP} = -\tilde{z}^\alpha_{\rm us} \ , \quad \tilde{z}^\alpha_{\rm FP} = -z^\alpha_{\rm us} \ , \tag{6.36}$$

these are precisely Freedman-Pufu's BPS conditions in [1]. So, we've recovered their setup.

UV expansion

Fefferman-Graham expansion of the metric:

$$ds^2 = L^2\left(d\rho^2 + e^{2A} ds^2_{S^3}\right) \ . \tag{6.37}$$

The fields have the following expansions:

$$\begin{aligned} e^A &= e^\rho\left(a_0 + a_1 e^{-\rho} + \ldots\right) \ , \\ z^\alpha &= e^{-\rho}\left(z_0^\alpha + z_1^\alpha e^{-\rho} + \ldots\right) \ , \\ \tilde{z}^\alpha &= e^{-\rho}\left(\tilde{z}_0^\alpha + \tilde{z}_1^\alpha e^{-\rho} + \ldots\right) \ . \end{aligned} \tag{6.38}$$

Note that a_0 can be set to take any value we want via a shift of the coordinate ρ. Plugging this expansion into the BPS conditions, we find no constraints at leading order. At subleading order, though, we find that:

$$a_1 = 0 \ , \tag{6.39}$$

as well as

$$\begin{aligned} z_1^\alpha &= -\frac{\tilde{z}_0^1 \tilde{z}_0^2 \tilde{z}_0^3}{\tilde{z}_0^\alpha} + \xi \frac{z_0^\alpha}{a_0} \ , \\ \tilde{z}_1^\alpha &= -\frac{z_0^1 z_0^2 z_0^3}{z_0^\alpha} - \xi \frac{\tilde{z}_0^\alpha}{a_0} \ . \end{aligned} \tag{6.40}$$

IR expansion

In the bulk of our spacetime, it will be convenient to use the conformal gauge for our metric ansatz:

$$ds^2 = \frac{L^2}{r^2} e^{2A}\left(dr^2 + r^2 ds^2_{S^3}\right) \ , \tag{6.41}$$

which is equivalent to picking our radial coordinate r such that

$$e^B = \frac{L}{r} e^A \ . \tag{6.42}$$

We are interested in bulk solutions that cap off smoothly by having the size of the S^3 go to zero at some point in the bulk. In order for the conformal gauge metric to be smooth at $r = 0$, the metric function must have an expansion of the form

$$e^A = r + \mathcal{O}(r^2) \ . \tag{6.43}$$

Additionally, we will also assume the scalar fields are regular and take non-singular values at $r = 0$. Plugging all of this into the BPS equations, we find two branches of solutions depending on the value of ξ:

$$\begin{aligned} &\text{Branch 1}: \quad \xi = -1 \ , \quad z^\alpha(0)\tilde{z}^\alpha(0) = z^1(0) z^2(0) z^3(0) \ , \\ &\text{Branch 2}: \quad \xi = 1 \ , \quad z^\alpha(0)\tilde{z}^\alpha(0) = \tilde{z}^1(0)\tilde{z}^2(0)\tilde{z}^3(0) \ . \end{aligned} \tag{6.44}$$

In the terminology of [1], Branch 1 is the $OSp(2|2)_L \times SU(2)_R$ branch, while Branch 2 is the $SU(2)_L \times OSp(2|2)_R$ branch. These labels refer to the representations of the Killing spinors, which we have not gone into detail about here. Morally, one should think of Branch 1 as fixing the IR values of $\tilde{z}^\alpha$, while Branch 2 fixes the IR values of z^α.

Full Solution

A careful analysis of the BPS equations (6.35) shows that the most general BPS solution in the conformal gauge (6.41) is as follows:

$$
\begin{aligned}
\text{Branch 1}: \quad & ds^2 = \frac{4L^2(1+c_1c_2c_3)(1+c_1c_2c_3r^4)}{(1-r^2)^2(1+c_1c_2c_3r^2)^2}\left(dr^2+r^2d\Omega_3^2\right) \ , \\
& z^\alpha = -c_\alpha \left(\frac{1-r^2}{1+c_1c_2c_3r^2}\right) \ , \\
& \tilde{z}^\alpha = \frac{c_1c_2c_3}{c_\alpha}\left(\frac{1-r^2}{1+c_1c_2c_3r^2}\right) \ , \\
\text{Branch 2}: \quad & ds^2 = \frac{4L^2(1+c_1c_2c_3)(1+c_1c_2c_3r^4)}{(1-r^2)^2(1+c_1c_2c_3r^2)^2}\left(dr^2+r^2d\Omega_3^2\right) \ , \\
& z^\alpha = \frac{c_1c_2c_3}{c_\alpha}\left(\frac{1-r^2}{1+c_1c_2c_3r^2}\right) \ , \\
& \tilde{z}^\alpha = -c_\alpha \left(\frac{1-r^2}{1+c_1c_2c_3r^2}\right) \ ,
\end{aligned}
\tag{6.45}
$$

where c_α are complex numbers chosen such that $c_1c_2c_3 \in \mathbb{R}$ and $|z^\alpha| < 1$, $|\tilde{z}^\alpha| < 1$ along the entire flow. (The former condition comes from wanting a real slicing of the metric, while the latter is just the condition that the scalars parameterize coordinates on the Poincaré disk.) Note that we have chosen the normalization on the scalars such that the constants c_α in these expressions are exactly the same as in [1]. The functions A and B in our ansatz (6.1) are hence given by

$$
\begin{aligned}
e^{2A} &= \frac{4r^2(1+c_1c_2c_3)(1+c_1c_2c_3r^4)}{(1-r^2)^2(1+c_1c_2c_3r^2)^2} \ , \\
e^{2B} &= \frac{L^2}{r^2}e^{2A} = \frac{4L^2(1+c_1c_2c_3)(1+c_1c_2c_3r^4)}{(1-r^2)^2(1+c_1c_2c_3r^2)^2} \ .
\end{aligned}
\tag{6.46}
$$

Note that we chose the coordinate frame such that the radial coordinate takes values on $r \in [0, 1)$, where the IR cap-off is at $r = 0$ and the UV boundary is at $r = 1$.

It may additionally be useful to note the form of the projectors M and $\widetilde{M}$ for the above solutions:

$$
\begin{aligned}
\text{Branch 1}: \quad & M = \widetilde{M}^{-1} = \frac{1}{r}\sqrt{\frac{(1+c_1c_2c_3)(1+c_1c_2c_3r^4)}{(1+c_1c_2c_3r^2)^2+c_1c_2c_3(1-r^2)^2}} \ , \\
\text{Branch 2}: \quad & \widetilde{M} = M^{-1} = \frac{1}{r}\sqrt{\frac{(1+c_1c_2c_3)(1+c_1c_2c_3r^4)}{(1+c_1c_2c_3r^2)^2+c_1c_2c_3(1-r^2)^2}} \ .
\end{aligned}
\tag{6.47}
$$

Notice that in the IR limit where $r \to 0$, we find that $\widetilde{M} \to 0$ on Branch 1, while $M \to 0$ on Branch 2. This is compatible with the IR boundary conditions prescribed by the full set of BPS equations, which demand that one of the projectors must vanish in the IR.

6.4 Holographic renormalization

We now want to evaluate the on-shell action. When we do so the result will be divergent as $r \to 1$, so we will take the UV boundary to be at a finite radial coordinate $0 < r_b < 1$, add the appropriate counterterms at this cut-off surface, and then take the limit where $r_b \to 1$ after things have been regularized.

Taking the bulk action (4.3) on our ansatz (6.1) and using the equations of motion, we find that

$$S_{\text{bulk}} = \frac{\pi L^3}{4G_N} \int_0^{r_b} dr \left[-\frac{2e^{A+B}}{L^2} + (e^{3A-B} A')' \right] . \tag{6.48}$$

Note that this cannot be written as a total derivative. I hoped this is because our ansatz is too restrictive; we should allow for general squashings of the boundary and turn on gauge fields, evaluate the on-shell action, and only after that implement the more restricted ansatz. This is investigated (with a seemingly null result) in Appendix B.

The counterterm action is

$$S_{\text{CT}} = S_{\text{GH}} + S_{\mathcal{R}} + S_{\text{SUSY}} , \tag{6.49}$$

with

$$\begin{aligned} S_{\text{GH}} &\equiv -\frac{1}{8\pi G_N} \int d^3x \sqrt{h}\, K , \\ S_{\mathcal{R}} &\equiv \frac{L}{16\pi G_N} \int d^3x \sqrt{h}\, \mathcal{R} , \\ S_{\text{SUSY}} &\equiv \frac{1}{8\pi G_N L} \int d^3x \sqrt{h}\, e^{\mathcal{K}/2} \sqrt{\mathcal{V}\tilde{\mathcal{V}}} , \end{aligned} \tag{6.50}$$

where h is the determinant of the induced metric, h_{ab}, on the conformal boundary, K is the trace of the extrinsic curvature, $\mathcal{R}$ the Ricci scalar of h_{ab}, and $\mathcal{V}$ and $\tilde{\mathcal{V}}$ are the superpotentials

$$\mathcal{V} = 2(z^1 z^2 z^3 - 1) , \quad \tilde{\mathcal{V}} = 2(\tilde{z}^1 \tilde{z}^2 \tilde{z}^3 - 1) . \tag{6.51}$$

All of these are UV counterterms and should be evaluated at the cut-off surface $r = r_b$. On our ansatz, the counterterm action is

$$S_{\text{CT}} = \frac{\pi L^2}{4G_N} \left[-3Le^{3A-B} A' + 3e^A + 2e^{3A} \sqrt{\frac{(1 - z^1 z^2 z^3)(1 - \tilde{z}^1 \tilde{z}^2 \tilde{z}^3)}{(1 - z^1 \tilde{z}^1)(1 - z^2 \tilde{z}^2)(1 - z^3 \tilde{z}^3)}} \right]_{r=r_b} . \tag{6.52}$$

The full regularized on-shell action is simply

$$S_{\text{on-shell}} = S_{\text{bulk}} + S_{\text{CT}} . \tag{6.53}$$

If we now plug in the Freedman-Pufu solution explicitly, evaluate at $r = r_b < 1$, and then take the limit $r_b \to 1$, we find that

$$S_{\text{on-shell}} = \frac{\pi L^2}{2G_N} \left(\frac{1 - c_1 c_2 c_3}{1 + c_1 c_2 c_3} \right) . \tag{6.54}$$

Alternative Quantization

Following the analysis of [1], one can show via the full $\mathcal{N} = 8$ theory that the linear combinations $z^\alpha + \tilde{z}^\alpha$ transform as scalars while $z^\alpha - \tilde{z}^\alpha$ transform as pseudoscalars. All pseudoscalars are quantized with regular boundary conditions, while the scalars obey alternative quantization and must satisfy the alternate boundary conditions prescribed in [8].

If we were to take the usual standard boundary conditions, then in the Fefferman-Graham expansion of the scalars

$$z^\alpha = z_0^\alpha e^{-|\rho|} + z_1^\alpha e^{-2|\rho|} + \mathcal{O}\left(e^{-3|\rho|}\right) \ , \tag{6.55}$$

we would interpret the leading piece z_0^α as determining the source and the subleading piece z_1^α the VEV in the dual field theory. The on-shell action is then a function of the sources z_0^α and $\tilde{z}_0^\alpha$ and it should match the partition function of the dual field theory. However, for alternative quantization the role of the source and the VEV are interchanged; the source is the canonical conjugate to z_0^α while the VEV is the canonical conjugate to z_1^α. The on-shell action must thus be Legendre-transformed with respect to the scalars $z^\alpha + \tilde{z}^\alpha$ in order for it to be a function of only the sources. This Legendre-transformed action (denoted by J) is then what gets identified with the partition function, i.e. $J = -\log Z$.

In order to implement the Legendre transform, we first need the canonical conjugates Π^α and $\tilde{\Pi}^\alpha$ of z^α and $\tilde{z}^\alpha$, respectively, defined by

$$\Pi^\alpha \equiv \frac{\delta S_{\text{on-shell}}}{\delta(\partial_\rho z^\alpha)} \ , \quad \tilde{\Pi}^\alpha \equiv \frac{\delta S_{\text{on-shell}}}{\delta(\partial_\rho \tilde{z}^\alpha)} \ , \tag{6.56}$$

where ρ is the radial Fefferman-Graham coordinate. For the specific Fefferman-Graham expansion (6.38) for our solutions with an S^3 boundary, we compute explicitly that these are given by

$$\begin{aligned} \Pi^\alpha &= -\frac{a_0^3 L^2 e^\rho}{8\pi G_N}\left(z_1^\alpha + \frac{\tilde{z}_0^1 \tilde{z}_0^2 \tilde{z}_0^3}{\tilde{z}_0^\alpha}\right) + \dots \ , \\ \tilde{\Pi}^\alpha &= -\frac{a_0^3 L^2 e^\rho}{8\pi G_N}\left(\tilde{z}_1^\alpha + \frac{z_0^1 z_0^2 z_0^3}{z_0^\alpha}\right) + \dots \ , \end{aligned} \tag{6.57}$$

where the dots indicate terms that are finite as $\rho \to \infty$. Once we implement the BPS conditions, these become

$$\begin{aligned} \Pi^\alpha &= -\xi \frac{a_0^2 z_0^\alpha L^2 e^\rho}{8\pi G_N} + \dots \ , \\ \tilde{\Pi}^\alpha &= \xi \frac{a_0^2 \tilde{z}_0^\alpha L^2 e^\rho}{8\pi G_N} + \dots \ . \end{aligned} \tag{6.58}$$

Then, since $z^\alpha - \tilde{z}^\alpha$ are pseudoscalars that obey standard quantization while $z^\alpha + \tilde{z}^\alpha$ are scalars that obey alternate quantization, we find the following expressions at the boundary for the sources σ_F^α and D_F^α for the pseudoscalars and scalars, respectively:

$$\begin{aligned} \sigma_F^\alpha &\propto \lim_{\rho\to\infty} e^\rho (z^\alpha - \tilde{z}^\alpha) = z_0^\alpha - \tilde{z}_0^\alpha \ , \\ D_F^\alpha &\propto \lim_{\rho\to\infty} e^{-\rho}(\Pi^\alpha + \tilde{\Pi}^\alpha) = -\xi \frac{a_0^2 L^2}{8\pi G_N}\left(z_0^\alpha - \tilde{z}_0^\alpha\right) \ . \end{aligned} \tag{6.59}$$

Note that these source expressions are only true up to some as-of-yet undetermined proportionality constant. Nonetheless, these relations automatically tell us that $\sigma_F^\alpha \propto D_F^\alpha$, as required by supersymmetry in the field theory [1]. That is, the asymptotic expansion is enough to validate the Ward identities; we do not need a full bulk solution.

To actually compute these sources for our explicit solutions, we first recall that the coordinate r in conformal gauge is related to the the Fefferman-Graham coordinate ρ by the condition that

$$e^{2B} dr^2 = L^2 d\rho^2 \ . \tag{6.60}$$

This condition leads to a series expansion of the form

$$r = 1 - \alpha e^{-\rho} + \frac{\alpha^2}{2} e^{-2\rho} - \frac{\alpha^3}{4} \left(\frac{1 - c_1 c_2 c_3}{1 + c_1 c_2 c_3} \right)^2 e^{-3\rho} + \dots \ , \tag{6.61}$$

for some constant α. In turn, this means that the UV expansion for the metric function A takes the form

$$e^{2A} = e^\rho \left(\frac{1}{\alpha} + \dots \right) \ , \tag{6.62}$$

which upon comparison with the Fefferman-Graham expansion (6.38) means we should identify $\alpha = 1/a_0$. Armed with this, we can now write down the UV expansion of the scalars:

$$\begin{aligned}
\text{Branch 1:} \quad z^\alpha &= e^{-\rho} \left(- \frac{2 c_\alpha}{a_0 (1 + c_1 c_2 c_3)} + \frac{2 c_\alpha (1 - c_1 c_2 c_3)}{a_0^2 (1 + c_1 c_2 c_3)^2} + \dots \right) \ , \\
\tilde{z}^\alpha &= e^{-\rho} \left(\frac{2 c_1 c_2 c_3}{a_0 c_\alpha (1 + c_1 c_2 c_3)} - \frac{2 c_1 c_2 c_3 (1 - c_1 c_2 c_3)}{a_0^2 c_\alpha (1 + c_1 c_2 c_3)^2} e^{-\rho} + \dots \right) \ , \\
\text{Branch 2:} \quad z^\alpha &= e^{-\rho} \left(\frac{2 c_1 c_2 c_3}{a_0 c_\alpha (1 + c_1 c_2 c_3)} - \frac{2 c_1 c_2 c_3 (1 - c_1 c_2 c_3)}{a_0^2 c_\alpha (1 + c_1 c_2 c_3)^2} e^{-\rho} + \dots \right) \ , \\
\tilde{z}^\alpha &= e^{-\rho} \left(- \frac{2 c_\alpha}{a_0 (1 + c_1 c_2 c_3)} + \frac{2 c_\alpha (1 - c_1 c_2 c_3)}{a_0^2 (1 + c_1 c_2 c_3)^2} + \dots \right) \ .
\end{aligned} \tag{6.63}$$

Comparing to the Fefferman-Graham expansion (6.38), we find that:

$$\begin{aligned}
\text{Branch 1:} \quad z_0^\alpha &= - \frac{2 c_\alpha}{a_0 (1 + c_1 c_2 c_3)} \ , \qquad \tilde{z}_0^\alpha = \frac{2 c_1 c_2 c_3}{a_0 c_\alpha (1 + c_1 c_2 c_3)} \ , \\
\text{Branch 2:} \quad z_0^\alpha &= \frac{2 c_1 c_2 c_3}{a_0 c_\alpha (1 + c_1 c_2 c_3)} \ , \qquad \tilde{z}_0^\alpha = - \frac{2 c_\alpha}{a_0 (1 + c_1 c_2 c_3)} \ .
\end{aligned} \tag{6.64}$$

Note that we go between branches simply by swapping $z^\alpha \leftrightarrow \tilde{z}^\alpha$. The source relations are therefore given by:

$$\begin{aligned}
\sigma_F^\alpha &\propto \frac{2\xi}{a_0 (1 + c_1 c_2 c_3)} \left(c_\alpha + \frac{c_1 c_2 c_3}{c_\alpha} \right) \ , \\
D_F^\alpha &\propto - \frac{2 a_0 L^2}{8 \pi G_N (1 + c_1 c_2 c_3)} \left(c_\alpha + \frac{c_1 c_2 c_3}{c_\alpha} \right) \ ,
\end{aligned} \tag{6.65}$$

where $\xi = -1$ on Branch 1 and $\xi = 1$ on Branch 2. Once again, these source relations are only up to some overall undetermined constants.

The Legendre-transformed on-shell action can now be succinctly written down as

$$J = S_{\text{on-shell}} - \frac{1}{2}\sum_{\alpha=1}^{3}\int_{\text{UV}} d^3x\,(z^\alpha + \tilde{z}^\alpha)(\Pi^\alpha + \tilde{\Pi}^\alpha)\ , \tag{6.66}$$

where the integration is over the coordinates on the UV conformal boundary. Plugging everything in, we find that:

$$J = \frac{\pi L^2}{2G_N}\frac{(1-c_1^2)(1-c_2^2)(1-c_3^2)}{(1+c_1c_2c_3)^2}\ , \tag{6.67}$$

irrespective of the branch.

6.5 Field theory matching

To summarize the above section, the gravitational partition function is given by

$$\log Z_{\text{grav}} = -J = -\frac{\pi L^2}{2G_N}\frac{(1-c_1^2)(1-c_2^2)(1-c_3^2)}{(1+c_1c_2c_3)^2}\ . \tag{6.68}$$

We can make this expression a little more enlightening by taking inspiration from the UV/IR correspondence in [2]. We will define the quantities $\hat{u}^I$ by

$$\hat{u}^I = \begin{cases} \left.\dfrac{2\pi X^I}{\sum_J X^J}\right|_{\text{IR}} & \text{on Branch 1}\ ,\\[2ex] \left.\dfrac{2\pi \bar{X}^I}{\sum_J \bar{X}^J}\right|_{\text{IR}} & \text{on Branch 2}\ ,\end{cases} \tag{6.69}$$

such that the $\hat{u}^I$ are determined entirely by the IR values of the scalars. Explicitly, they are given by

$$\begin{aligned} \hat{u}^0 &= \frac{\pi(1+c_1)(1+c_2)(1+c_3)}{1+c_1c_2c_3}\ ,\\ \hat{u}^1 &= \frac{\pi(1+c_1)(1-c_2)(1-c_3)}{1+c_1c_2c_3}\ ,\\ \hat{u}^2 &= \frac{\pi(1-c_1)(1+c_2)(1-c_3)}{1+c_1c_2c_3}\ ,\\ \hat{u}^3 &= \frac{\pi(1-c_1)(1-c_2)(1+c_3)}{1+c_1c_2c_3}\ . \end{aligned} \tag{6.70}$$

One can then show that the gravitational partition function (6.68) can be expressed more compactly as

$$\log Z_{\text{grav}} = -\frac{L^2}{2\pi G_N}\sqrt{\hat{u}^0\hat{u}^1\hat{u}^2\hat{u}^3}\ . \tag{6.71}$$

In the dual ABJM theory, with gauge group $U(N)_k \times U(N)_{-k}$ at Chern-Simons level $k=1$, the partition function is

$$\log Z_{\text{CFT}} = -\frac{4\sqrt{2}\pi N^{3/2}}{3}\sqrt{R[Z^1]R[Z^2]R[W_1]R[W_2]}\ , \tag{6.72}$$

where Z^a and W_a are the bifundamental chiral multiplets that transform in the $(\bar{\mathbf{N}}, \mathbf{N})$ and $(\mathbf{N}, \bar{\mathbf{N}})$ representations of the gauge group, respectively, and R represents their associated R-charges. These R-charges can be written as:

$$\begin{aligned} R[Z^1] &= \frac{1}{2} \pm ia\left(\sigma_F^1 + \sigma_F^2 + \sigma_F^3\right) \ , \\ R[Z^2] &= \frac{1}{2} \pm ia\left(\sigma_F^1 - \sigma_F^2 - \sigma_F^3\right) \ , \\ R[W_1] &= \frac{1}{2} \pm ia\left(-\sigma_F^1 + \sigma_F^2 - \sigma_F^3\right) \ , \\ R[W_2] &= \frac{1}{2} \pm ia\left(-\sigma_F^1 - \sigma_F^2 + \sigma_F^3\right) \ , \end{aligned} \tag{6.73}$$

where the σ_F^α denote the three real masses associated with the three background flavor current multiplets. The Ward identities for ABJM on S^3 require the scalar sources to be related by $\sigma_F^\alpha = \pm ia D_F^\alpha$. This can be satisfied by setting

$$\sigma_F^\alpha = \mp i \frac{\delta_\alpha}{a} \ , \quad D_F^\alpha = -\frac{\delta_\alpha}{a^2} \ , \tag{6.74}$$

for some constant of proportionality δ_α. If we compare the CFT scalar sources (6.74) to the gravity predictions for the sources (6.65), we find that we must set the δ_α to be given by

$$\delta_\alpha = \frac{n}{1 + c_1 c_2 c_3}\left(c_\alpha + \frac{c_1 c_2 c_3}{c_\alpha}\right) \ , \tag{6.75}$$

up to some overall normalization n. The R-charges of the chiral multiplets are therefore given in terms of the constants c_α by

$$\begin{aligned} R[Z^1] &= \frac{1 + c_1c_2c_3 + 2n\left(c_1 + c_2 + c_3 + c_1c_2 + c_2c_3 + c_3c_1\right)}{2(1 + c_1c_2c_3)} \ , \\ R[Z^2] &= \frac{1 + c_1c_2c_3 + 2n\left(c_1 - c_2 - c_3 - c_1c_2 + c_2c_3 - c_3c_1\right)}{2(1 + c_1c_2c_3)} \ , \\ R[W_1] &= \frac{1 + c_1c_2c_3 + 2n\left(-c_1 + c_2 - c_3 - c_1c_2 - c_2c_3 + c_3c_1\right)}{2(1 + c_1c_2c_3)} \ , \\ R[W_2] &= \frac{1 + c_1c_2c_3 + 2n\left(-c_1 - c_2 + c_3 + c_1c_2 - c_2c_3 - c_3c_1\right)}{2(1 + c_1c_2c_3)} \ . \end{aligned} \tag{6.76}$$

Our goal now is to determine the normalization factor n by matching the gravitational partition function (6.71) to the field theory partition function (6.72). The usual AdS$_4$/CFT$_3$ dictionary relates the AdS radius L to the CFT gauge group rank N by

$$\frac{\sqrt{2} N^{3/2}}{3} = \frac{L^2}{2G_N} \ . \tag{6.77}$$

Armed with this, if we compare the partition functions (6.71) and (6.72) and use the relationship (6.75) between the field theory sources and the scalar fields in the bulk, we find that there is a match if and only if $n = 1/2$. Moreover, setting $n = 1/2$ also has the effect of making the R-charges related to the gravitational quantities $\hat{u}^I$ we defined earlier. In particular, we find that

$$\hat{u}^0 = 2\pi R[Z^1] \ , \quad \hat{u}^1 = 2\pi R[Z^2] \ , \quad \hat{u}^2 = 2\pi R[W_1] \ , \quad \hat{u}^3 = 2\pi R[W_2] \ . \tag{6.78}$$

This is similar in spirit to the match established in [2] between the bulk quantities $\hat{u}^I$ and the complex fugacities u^I in the field theory when placing ABJM on $S^1 \times \Sigma_{\mathfrak{g}}$. In the case at hand of S^3 there are no fugacities one can define, but the R-charges are related to the real masses so the match is nonetheless of a similar form. Note also that this is an example of a **UV/IR correspondence**, since the $\hat{u}^I$ are determined entirely by the IR values of the supergravity scalars, while the CFT sources can be calculated via the UV expansion of the bulk fields.

7 STU model with $SU(2) \times U(1)$ squashed S^3 boundary

In this section, our goal is to construct Euclidean BPS solutions in the STU model whose boundary is that of a squashed S^3. In particular, we'd like to find new solutions that have non-trivial scalar profiles. Once we have these solutions, we want to compute their on-shell action, and then in turn compute the partition function via a Legendre transform. This will lead us to a novel prediction for the partition function of ABJM on a squashed sphere with background sources turned on.

7.1 Ansatz and variations

The metric ansatz for having a squashed sphere at the boundary is

$$ds^2 = \frac{1}{4}L^2 \left[e^{2A(r)}\left(\sigma_1^2 + \sigma_2^2\right) + e^{2C(r)}\sigma_3^2\right] + e^{2B(r)}dr^2 \ , \tag{7.1}$$

for some functions A, B, and C of the radial coordinate r. The vielbein are

$$\begin{aligned} e^a &= \frac{1}{2}Le^A\sigma_a \ , \quad (a = 1,2) \\ e^3 &= \frac{1}{2}Le^C\sigma_3 \ , \\ e^4 &= e^B dr \ , \end{aligned} \tag{7.2}$$

and the spin connection components are

$$\begin{aligned} \omega^{12} &= \left(1 - \frac{1}{2}e^{2(C-A)}\right)\sigma_3 \ , \\ \omega^{23} &= \frac{1}{2}e^{C-A}\sigma_1 \ , \\ \omega^{31} &= \frac{1}{2}e^{C-A}\sigma_2 \ , \\ \omega^{a4} &= \frac{1}{2}Le^{A-B}A'\sigma_a \ , \quad (a = 1,2) \\ \omega^{34} &= \frac{1}{2}Le^{C-B}C'\sigma_3 \ . \end{aligned} \tag{7.3}$$

Since we have in mind that we are squashing along the σ_3 direction, we need to correspondingly turn on gauge fields along this direction as well:

$$A^I = e^I(r)\sigma_3 \ , \tag{7.4}$$

with e^I being some complex function of the radial coordinate, and so the field strengths are

$$F^I = \partial_r e^I dr \wedge \sigma_3 + e^I \sigma_1 \wedge \sigma_2 = -\frac{2}{L} e^{-B-C} \partial_r e^I dx^3 \wedge dx^4 + \frac{4}{L^2} e^{-2A} e^I dx^1 \wedge dx^2 \ . \tag{7.5}$$

It will be useful to note the (anti-)self-dual components:

$$F_{34}^{I\pm} = -\frac{1}{L} e^{-B-C} \partial_r e^I \pm \frac{2}{L^2} e^{-2A} e^I \ . \tag{7.6}$$

To make notation a little simpler, we will define

$$n_I^- = \mathcal{I}_{IJ} F_{34}^{-J} \ , \quad n_I^+ = \mathcal{I}_{IJ} (F_{34}^{+J})^* \ . \tag{7.7}$$

Then, the variations are

$$\begin{aligned}
\delta\psi_r^i &= \left(\partial_r + \frac{i}{2}\mathcal{A}_r\right)\epsilon^i + \frac{i}{2\sqrt{2}L} e^B P_I^3 L^I \gamma_4 (\sigma_3)^{ij}\epsilon_j + e^B L^I n_I^- \gamma_3 \varepsilon^{ij}\epsilon_j \ , \\
\delta\psi_{\sigma_1}^i &= \left(\partial_{\sigma_1} + \frac{1}{4} L e^{A-B} A' \gamma_{14} - \frac{1}{4} e^{C-A}\gamma_{14} + \frac{i}{2}\mathcal{A}_{\sigma_1}\right)\epsilon^i + \frac{i}{4\sqrt{2}} e^A P_I^3 L^I \gamma_1 (\sigma_3)^{ij}\epsilon_j \\
&\quad + \frac{1}{2} L e^A L^I n_I^- \gamma_{134} \varepsilon^{ij}\epsilon_j \ , \\
\delta\psi_{\sigma_2}^i &= \left(\partial_{\sigma_2} + \frac{1}{4} L e^{A-B} A' \gamma_{24} - \frac{1}{4} e^{C-A}\gamma_{24} + \frac{i}{2}\mathcal{A}_{\sigma_2}\right)\epsilon^i + \frac{i}{4\sqrt{2}} e^A P_I^3 L^I \gamma_2 (\sigma_3)^{ij}\epsilon_j \\
&\quad + \frac{1}{2} L e^A L^I n_I^- \gamma_{234} \varepsilon^{ij}\epsilon_j \ , \\
\delta\psi_{\sigma_3}^i &= \left(\partial_{\sigma_3} + \frac{1}{4} L e^{C-B} C' \gamma_{34} + \frac{1}{4}(e^{2(C-A)} - 2)\gamma_{34} + \frac{i}{2}\mathcal{A}_{\sigma_3}\right)\epsilon^i + \frac{i}{2\sqrt{2}L} P_I^3 e^I (\sigma_3)_j{}^i \epsilon^j \\
&\quad + \frac{i}{4\sqrt{2}} e^C P_I^3 L^I \gamma_3 (\sigma_3)^{ij}\epsilon_j - \frac{1}{2} L e^C L^I n_I^- \gamma_4 \varepsilon^{ij}\epsilon_j \ , \\
\delta\lambda_i^\alpha &= -2 g^{\alpha\bar\beta} f_{\bar\beta}^I n_I^- \gamma_{34}\varepsilon_{ij}\epsilon^j + e^{-B}(\partial_r z^\alpha)\gamma_4\epsilon_i + \frac{2}{L} e^{-C}(\partial_{\sigma_3} z^\alpha)\gamma_3\epsilon_i \\
&\quad + \frac{2}{L} e^{-A}(\partial_{\sigma_1} z^\alpha)\gamma_1\epsilon_i + \frac{2}{L} e^{-A}(\partial_{\sigma_2} z^\alpha)\gamma_2\epsilon_i - \frac{i}{\sqrt{2}L} g^{\alpha\bar\beta} f_{\bar\beta}^I P_I^3 (\sigma_3)_{ij}\epsilon^j \ ,
\end{aligned} \tag{7.8}$$

plus the conjugate variations

$$\begin{aligned}
\delta\psi_{ir} &= \left(\partial_r - \frac{i}{2}\mathcal{A}_r\right)\epsilon_i + \frac{i}{2\sqrt{2}L}e^B P_I^3 \bar{L}^I \gamma_4(\sigma_3)_{ij}\epsilon^j + e^B \bar{L}^I n_I^+ \gamma_3 \varepsilon_{ij}\epsilon^j \ , \\
\delta\psi_{i\sigma_1} &= \left(\partial_{\sigma_1} + \frac{1}{4}Le^{A-B}A'\gamma_{14} + \frac{1}{4}e^{C-A}\gamma_{14} - \frac{i}{2}\mathcal{A}^*_{\sigma_1}\right)\epsilon_i + \frac{i}{4\sqrt{2}}e^A P_I^3 \bar{L}^I \gamma_1(\sigma_3)_{ij}\epsilon^j \\
&\quad + \frac{1}{2}Le^A \bar{L}^I n_I^+ \gamma_{134}\varepsilon_{ij}\epsilon^j \ , \\
\delta\psi_{i\sigma_2} &= \left(\partial_{\sigma_2} + \frac{1}{4}Le^{A-B}A'\gamma_{24} + \frac{1}{4}e^{C-A}\gamma_{24} - \frac{i}{2}\mathcal{A}^*_{\sigma_2}\right)\epsilon_i + \frac{i}{4\sqrt{2}}e^A P_I^3 \bar{L}^I \gamma_2(\sigma_3)_{ij}\epsilon^j \\
&\quad + \frac{1}{2}Le^A \bar{L}^I n_I^+ \gamma_{234}\varepsilon_{ij}\epsilon^j \ , \\
\delta\psi_{i\sigma_3} &= \left(\partial_{\sigma_3} + \frac{1}{4}Le^{C-B}C'\gamma_{34} - \frac{1}{4}(e^{2(C-A)} - 2)\gamma_{34} - \frac{i}{2}\mathcal{A}^*_{\sigma_3}\right)\epsilon_i - \frac{i}{2\sqrt{2}L}P_I^3 (e^I)^*(\sigma_3)_i{}^j\epsilon_j \\
&\quad + \frac{i}{4\sqrt{2}}e^C P_I^3 \bar{L}^I \gamma_3(\sigma_3)_{ij}\epsilon^j - \frac{1}{2}Le^C \bar{L}^I n_I^+ \gamma_4 \varepsilon_{ij}\epsilon^j \ , \\
\delta\lambda^{i\bar{\alpha}} &= -2g^{\bar{\alpha}\beta} f_\beta^I n_I^+ \gamma_{34}\varepsilon^{ij}\epsilon_j + e^{-B}(\partial_r \tilde{z}^{\bar{\alpha}})\gamma_4\epsilon^i + \frac{2}{L}e^{-C}(\partial_{\sigma_3}\tilde{z}^{\bar{\alpha}})\gamma_3\epsilon^i \\
&\quad + \frac{2}{L}e^{-A}(\partial_{\sigma_1}\tilde{z}^{\bar{\alpha}})\gamma_1\epsilon^i + \frac{2}{L}e^{-A}(\partial_{\sigma_2}\tilde{z}^{\bar{\alpha}})\gamma_2\epsilon^i - \frac{i}{\sqrt{2}L}g^{\bar{\alpha}\beta} f_\beta^I P_I^3 (\sigma_3)^{ij}\epsilon_j \ .
\end{aligned} \tag{7.9}$$

7.2 Projectors and further ansatz

We will once again impose

$$\begin{aligned}
\gamma_4\epsilon_i &= iM(r)(\sigma_3)_{ij}\epsilon^j \ , \\
\gamma_4\epsilon^i &= i\widetilde{M}(r)(\sigma_3)^{ij}\epsilon_j \ ,
\end{aligned} \tag{7.10}$$

where M and $\widetilde{M}$ are some functions that satisfy $M\widetilde{M} = 1$. However, we break more supersymmetry by squashing the asymptotic S^3, and have to impose an additional projector:

$$\begin{aligned}
\gamma_3\epsilon_i &= N(r)\varepsilon_{ij}\epsilon^j \ , \\
\gamma_3\epsilon^i &= \widetilde{N}(r)\varepsilon^{ij}\epsilon_j \ ,
\end{aligned} \tag{7.11}$$

where $N\widetilde{N} = -1$. Note that consistency of the two sets of projectors requires $M\widetilde{N} = -\widetilde{M}N$, which in turn sets

$$\begin{aligned}
N &= \zeta M \ , \\
\widetilde{N} &= -\zeta\widetilde{M} \ ,
\end{aligned} \tag{7.12}$$

for some constant ζ that takes values $\zeta = \pm 1$. We will also assume that the scalars have vanishing derivative along the σ_i directions. The variations become:

$$
\begin{aligned}
\delta\psi^i_r &= \left(\partial_r + \frac{i}{2}\mathcal{A}_r + \frac{1}{2\sqrt{2}L}Me^B P^3_I L^I - \zeta M e^B L^I n^-_I\right)\epsilon^i \ , \\
\delta\psi^i_{\sigma_1} &= \partial_{\sigma_1}\epsilon^i + \frac{i}{4}\left(\widetilde{M}Le^{A-B}A' - \widetilde{M}e^{C-A} + \frac{1}{\sqrt{2}}e^A P^3_I L^I + 2\zeta L e^A L^I n^-_I\right)\gamma_1(\sigma_3)^{ij}\epsilon_j \ , \\
\delta\psi^i_{\sigma_2} &= \partial_{\sigma_2}\epsilon^i + \frac{i}{4}\left(\widetilde{M}Le^{A-B}A' - \widetilde{M}e^{C-A} + \frac{1}{\sqrt{2}}e^A P^3_I L^I + 2\zeta L e^A L^I n^-_I\right)\gamma_2(\sigma_3)^{ij}\epsilon_j \ , \\
\delta\psi^i_{\sigma_3} &= \partial_{\sigma_3}\epsilon^i + \frac{i}{4}\Bigg(\zeta L e^{C-B}C' - 2\zeta + \zeta e^{2(C-A)} + \frac{\sqrt{2}}{L}P^3_I e^I \\
&\qquad + \frac{1}{\sqrt{2}}\zeta M e^C P^3_I L^I - 2MLe^C L^I n^-_I\Bigg)(\sigma_3)_j{}^i\epsilon^j \ , \\
\delta\lambda^\alpha_i &= i\left(2\zeta g^{\alpha\bar\beta}f^I_{\bar\beta}n^-_I + Me^{-B}\partial_r z^\alpha - \frac{1}{\sqrt{2}L}g^{\alpha\bar\beta}f^I_{\bar\beta}P^3_I\right)(\sigma_3)_{ij}\epsilon^j \ ,
\end{aligned}
\tag{7.13}
$$

and the conjugate variations are given by very similar looking expressions:

$$
\begin{aligned}
\delta\psi_{ir} &= \left(\partial_r - \frac{i}{2}\mathcal{A}_r + \frac{1}{2\sqrt{2}L}\widetilde{M}e^B P^3_I \bar{L}^I + \zeta\widetilde{M}e^B\bar{L}^I n^+_I\right)\epsilon_i \ , \\
\delta\psi_{i\sigma_1} &= \partial_{\sigma_1}\epsilon_i + \frac{i}{4}\left(MLe^{A-B}A' + Me^{C-A} + \frac{1}{\sqrt{2}}e^A P^3_I\bar{L}^I - 2\zeta Le^A\bar{L}^I n^+_I\right)\gamma_1(\sigma_3)_{ij}\epsilon^j \ , \\
\delta\psi_{i\sigma_2} &= \partial_{\sigma_2}\epsilon_i + \frac{i}{4}\left(MLe^{A-B}A' + Me^{C-A} + \frac{1}{\sqrt{2}}e^A P^3_I\bar{L}^I - 2\zeta Le^A\bar{L}^I n^+_I\right)\gamma_2(\sigma_3)_{ij}\epsilon^j \ , \\
\delta\psi_{i\sigma_3} &= \partial_{\sigma_3}\epsilon_i + \frac{i}{4}\Bigg(\zeta Le^{C-B}C' + 2\zeta - \zeta e^{2(C-A)} - \frac{\sqrt{2}}{L}P^3_I(e^I)^* \\
&\qquad + \frac{1}{\sqrt{2}}\zeta\widetilde{M}e^C P^3_I\bar{L}^I + 2\widetilde{M}Le^C\bar{L}^I n^+_I\Bigg)(\sigma_3)_i{}^j\epsilon_j \ , \\
\delta\lambda^{i\bar\alpha} &= i\left(-2\zeta g^{\bar\alpha\beta}f^I_\beta n^+_I + \widetilde{M}e^{-B}\partial_r\tilde{z}^{\bar\alpha} - \frac{1}{\sqrt{2}L}g^{\bar\alpha\beta}f^I_\beta P^3_I\right)(\sigma_3)^{ij}\epsilon_j \ .
\end{aligned}
\tag{7.14}
$$

Note that the σ_3 component of the gravitino variation takes a somewhat form than the others, due to the gauge fields along the σ_3 direction as well as due to the projector relations above.

7.3 BPS conditions

Setting all variations above to zero yields the BPS conditions we are after. The gaugino variations tell us that:

$$
\begin{aligned}
0 &= 2\zeta g^{\alpha\bar\beta}f^I_{\bar\beta}n^-_I + Me^{-B}\partial_r z^\alpha - \frac{1}{\sqrt{2}L}g^{\alpha\bar\beta}f^I_{\bar\beta}P^3_I \ , \\
0 &= -2\zeta g^{\bar\alpha\beta}f^I_\beta n^+_I + \widetilde{M}e^{-B}\partial_r\tilde{z}^{\bar\alpha} - \frac{1}{\sqrt{2}L}g^{\bar\alpha\beta}f^I_\beta P^3_I \ .
\end{aligned}
\tag{7.15}
$$

The radial components of the gravitino variations tell us the radial dependence of the supersymmetry spinor parameters:

$$
\begin{aligned}
0 &= \left(\partial_r + \frac{i}{2}\mathcal{A}_r + \frac{1}{2\sqrt{2}L} M e^B P_I^3 L^I - \zeta M e^B L^I n_I^- \right) \epsilon^i \ , \\
0 &= \left(\partial_r - \frac{i}{2}\mathcal{A}_r + \frac{1}{2\sqrt{2}L} \widetilde{M} e^B P_I^3 \bar{L}^I + \zeta \widetilde{M} e^B \bar{L}^I n_I^+ \right) \epsilon_i \ .
\end{aligned}
\tag{7.16}
$$

For the components of the gravitino variations along the S^3, we first define

$$
\begin{aligned}
\phi &= M L e^{A-B} A' + M e^{C-A} + \frac{1}{\sqrt{2}} e^A P_I^3 \bar{L}^I - 2\zeta L e^A \bar{L}^I n_I^+ \ , \\
\tilde{\phi} &= \widetilde{M} L e^{A-B} A' - \widetilde{M} e^{C-A} + \frac{1}{\sqrt{2}} e^A P_I^3 L^I + 2\zeta L e^A L^I n_I^- \ , \\
\chi &= \zeta L e^{C-B} C' + 2\zeta - \zeta e^{2(C-A)} - \frac{\sqrt{2}}{L} P_I^3 (e^I)^* + \frac{1}{\sqrt{2}} \zeta \widetilde{M} e^C P_I^3 \bar{L}^I + 2\widetilde{M} L e^C \bar{L}^I n_I^+ \ , \\
\tilde{\chi} &= \zeta L e^{C-B} C' - 2\zeta + \zeta e^{2(C-A)} + \frac{\sqrt{2}}{L} P_I^3 e^I + \frac{1}{\sqrt{2}} \zeta M e^C P_I^3 L^I - 2 M L e^C L^I n_I^- \ .
\end{aligned}
\tag{7.17}
$$

Then, the corresponding BPS conditions can be written as follows:

$$
\begin{aligned}
\partial_{\sigma_1} \epsilon^i &= -\frac{i}{4} \tilde{\phi} \gamma_1 (\sigma_3)^{ij} \epsilon_j \ , \quad & \partial_{\sigma_1} \epsilon_i &= -\frac{i}{4} \phi \gamma_1 (\sigma_3)_{ij} \epsilon^j \ , \\
\partial_{\sigma_2} \epsilon^i &= -\frac{i}{4} \tilde{\phi} \gamma_2 (\sigma_3)^{ij} \epsilon_j \ , \quad & \partial_{\sigma_2} \epsilon_i &= -\frac{i}{4} \phi \gamma_2 (\sigma_3)_{ij} \epsilon^j \ , \\
\partial_{\sigma_3} \epsilon^i &= -\frac{i}{4} \tilde{\chi} (\sigma_3)_j{}^i \epsilon^j \ , \quad & \partial_{\sigma_3} \epsilon_i &= -\frac{i}{4} \chi (\sigma_3)_i{}^j \epsilon_j \ .
\end{aligned}
\tag{7.18}
$$

The derivatives along the Mauren-Cartan forms satisfy $[\partial_{\sigma_a}, \partial_{\sigma_b}] = -\varepsilon^{abc}\partial_{\sigma_c}$, which when combined with the BPS conditions (7.18) yields the following integrability conditions:

$$
\begin{aligned}
0 &= \phi(\chi + \tilde{\chi}) + 4\zeta\phi \ , \\
0 &= \tilde{\phi}(\chi + \tilde{\chi}) - 4\zeta\tilde{\phi} \ , \\
0 &= 2\chi - \zeta\phi\tilde{\phi} \ , \\
0 &= 2\tilde{\chi} - \zeta\phi\tilde{\phi} \ .
\end{aligned}
\tag{7.19}
$$

In the round S^3 case we found that there were two distinct sets of solutions to the corresponding integrability conditions. In the case at hand, however, the extra projection conditions lead to only a single solution. In particular, only the trivial solution satisfies the integrability conditions:

$$
\phi = \tilde{\phi} = \chi = \tilde{\chi} = 0 \ . \tag{7.20}
$$

Finally, the last piece of information we need in order to find solutions is the Maxwell equation, which demands that

$$
0 = \frac{4}{L^2} e^{-2A+B+C} \mathcal{I}_{IJ} e^J - \frac{2i}{L} (\partial_r \mathcal{R}_{IJ}) e^J - \partial_r \left(e^{2A-B-C} \mathcal{I}_{IJ} \partial_r e^J \right) \ . \tag{7.21}
$$

7.4 UV Expansion

Fefferman-Graham gauge:

$$ds^2 = L^2 d\rho^2 + \frac{L^2}{4}\left(e^{2A}\left(\sigma_1^2+\sigma_2^2\right)+e^{2C}\sigma_3^2\right) \ . \tag{7.22}$$

Consistency of the Fefferman-Graham gauge with the BPS equations and the equations of motion dictates the following asymptotic expansions:

$$\begin{aligned}
e^A &= e^\rho\left(a_0 + a_1 e^{-\rho}+\ldots\right) \ , \\
e^C &= e^\rho\left(c_0 + c_1 e^{-\rho}+\ldots\right) \ , \\
M &= M_0 + M_1 e^{-\rho} + \ldots \ , \\
\widetilde{M} &= \widetilde{M}_0 + \widetilde{M}_1 e^{-\rho} + \ldots \ , \\
z^\alpha &= e^{-\rho}\left(z_0^\alpha + z_1^\alpha e^{-\rho}+\ldots\right) \ , \\
\tilde{z}^\alpha &= e^{-\rho}\left(\tilde{z}_0^\alpha + \tilde{z}_1^\alpha e^{-\rho}+\ldots\right) \ , \\
e^I &= e_0^I + e_1^I e^{-\rho} + \ldots \ ,
\end{aligned} \tag{7.23}$$

for some set of constants $\{a_n, c_n, M_n, \widetilde{M}_n, z_n^\alpha, \tilde{z}_n^\alpha, e_n^I\}$. **(Anthony: Need to now plug this FG expansion into the BPS equations and see what we learn.)**

7.5 IR Expansion

We denote the location of the IR to be at some radial coordinate $r = r_0$. Based on the structure of various NUT solutions in minimal supergravity, I think that there should be a gauge for the radial coordinate such that the metric functions admit the following asymptotic expansion in the IR:

$$\begin{aligned}
e^A &= \alpha_1(r-r_0) + \alpha_2(r-r_0)^2 + \ldots \ , \\
e^B &= \beta_0 + \beta_1(r-r_0) + \ldots \ , \\
e^C &= \gamma_1(r-r_0) + \gamma_2(r-r_0)^2 + \ldots \ ,
\end{aligned} \tag{7.24}$$

for some set of constants $\{\alpha_i, \beta_i, \gamma_i\}$, where the dots indicate terms that are higher-order in powers of $(r-r_0)$. Crucially, we have enforced that both e^A and e^C vanish as $r \to r_0$, while e^B is non-zero. This in turn means that the metric has the following asymptotic expansion:

$$ds^2 \to \beta_0^2 dr^2 + \frac{L^2}{4}(r-r_0)^2\left(\alpha_1^2\left(\sigma_1^2+\sigma_2^2\right)+\gamma_1^2\sigma_3^2\right) + \ldots \ . \tag{7.25}$$

If we neglect the terms that are subleading as $r \to r_0$, then the metric (7.25) is simply that of $\mathbb{R}^4$, albeit written in a slightly unusual set of coordinates where we think of $\mathbb{R}^4$ as a squashed S^3 fibered over a radial coordinate.

The only caveat is that we may additionally have to demand that

$$\alpha_1 = \gamma_1 = \frac{2\beta_0}{L} \ , \tag{7.26}$$

in order for this metric to be free of any conical singularities at the origin. This is the most restrictive way to ensure that our spacetime caps off smoothly as $\mathbb{R}^4$ in the IR; I'm not sure if there's a more lenient way to do it that has fewer constraints. In principle there is no reason from a purely gravitational path integral point of view to demand regularity, but for our holographic purposes regularity is important.

For the other fields in our theory, the IR boundary conditions they should satisfy are as follows:

$$\begin{aligned} z^\alpha &= \zeta_0^\alpha + \zeta_1^\alpha (r - r_0) + \ldots \ , \\ \tilde{z}^\alpha &= \tilde{\zeta}_0^\alpha + \tilde{\zeta}_1^\alpha (r - r_0) + \ldots \ , \end{aligned} \tag{7.27}$$

Finally, for the projectors M and $\widetilde{M}$, they should satisfy

$$M = m_1 (r - r_0) + m_2 (r - r_0)^2 + \ldots \ , \quad \widetilde{M} = M^{-1} \ , \tag{7.28}$$

or vice-versa. That is, one of them should die out linearly as $r \to r_0$ and the other should blow up in the same limit. **(Anthony: Now need to plug these expansions into the BPS equations and see what we learn.)**

7.6 Solutions

The simplest solutions I could find are the $\frac{1}{4}$-BPS $SU(2) \times U(1)$ squashed spheres in minimal supergravity, as detailed in Section 5 of [9]. There is the self-dual solution, given by

$$\begin{aligned} ds^2 &= \frac{r^2 - s^2}{\Omega(r)} dr^2 + \left(r^2 - s^2\right) \left(\sigma_1^2 + \sigma_2^2\right) + \frac{4 s^2 \Omega(r)}{r^2 - s^2} \sigma_3^2 \ , \\ \Omega(r) &= \frac{(r+s)^2 \left(L^2 + r^2 - 2rs - 3s^2\right)}{L^2} \ , \\ A^I &= -\frac{\zeta \left(L^2 - 4s^2\right)}{2\sqrt{2} L} \left(\frac{r+s}{r-s}\right) \sigma_3 \ , \\ z^\alpha &= \tilde{z}^\alpha = 0 \ , \\ M &= \widetilde{M}^{-1} = \sqrt{\frac{r^2 - 2rs - 3s^2 + L^2}{r^2 - s^2}} \ . \end{aligned} \tag{7.29}$$

There is also the anti-self-dual solution, given by

$$\begin{aligned} ds^2 &= \frac{r^2 - s^2}{\Omega(r)} dr^2 + \left(r^2 - s^2\right) \left(\sigma_1^2 + \sigma_2^2\right) + \frac{4 s^2 \Omega(r)}{r^2 - s^2} \sigma_3^2 \ , \\ \Omega(r) &= \frac{(r-s)^2 \left(L^2 + r^2 + 2rs - 3s^2\right)}{L^2} \ , \\ A^I &= -\frac{\zeta \left(L^2 - 4s^2\right)}{2\sqrt{2} L} \left(\frac{r-s}{r+s}\right) \sigma_3 \ , \\ z^\alpha &= \tilde{z}^\alpha = 0 \ , \\ M &= \widetilde{M}^{-1} = \sqrt{\frac{r^2 - s^2}{r^2 + 2rs - 3s^2 + L^2}} \ . \end{aligned} \tag{7.30}$$

The free parameters in these solutions are the constant s and the constant sign $\zeta = \pm 1$. For both solutions, the parameter s can without loss of generality be chosen such that $s > 0$. For the anti-self-dual solution, the coordinate r takes values on the range $[s, \infty)$, where

$r = s$ corresponds to the IR while $r \to \infty$ is the UV. For the self-dual solution, r lives on the range $[r_0, \infty)$, where the IR coordinate r_0 is determined by finding the outermost root of $\Omega(r)$. **(Anthony: Need to now progress to finding new solutions with scalars turned on, but this has proven somewhat difficult and involved.)**

7.7 What about other solutions?

Something important to note is that the BPS equations I have derived are consistent with the 1/4-BPS squashed sphere solution of [9], but the same reference also discusses a different 1/2-BPS squashed sphere solution in its Section 4. In the conventions we use in this paper, this 1/2-BPS solution should take the following form:

$$\begin{aligned} ds^2 &= \frac{r^2 - s^2}{\Omega(r)} dr^2 + \left(r^2 - s^2\right)\left(\sigma_1^2 + \sigma_2^2\right) + \frac{4s^2\Omega(r)}{r^2 - s^2}\sigma_3^2 \ , \\ \Omega(r) &= \frac{(r-s)^2\left(L^2 + r^2 + 2rs - 3s^2\right)}{L^2} \ , \\ A^I &= \pm\frac{s}{\sqrt{2}}\sqrt{\frac{4s^2}{L^2} - 1}\left(\frac{r-s}{r+s}\right)\sigma_3 \ , \\ z^\alpha &= \tilde{z}^\alpha = 0 \ . \end{aligned} \tag{7.31}$$

I have tried to see if this 1/2-BPS solution solves the BPS equations I have derived, but it seems not to. So, the projectors I have chosen in equations (7.10) and (7.11) are incompatible with the supersymmetries preserved by the 1/2-BPS solution. So, on my list of things to do is to derive a new set of BPS conditions compatible with this 1/2-BPS solution, so that I can then start to add scalars to it.

Crucially, the 1/4-BPS solution should be thought of as sort of a "trivial" squashing, while the 1/2-BPS solution is non-trivial. Correspondingly, their on-shell actions reflect this, where the 1/4-BPS on-shell action is independent of the squashing parameter s, while the 1/2-BPS on-shell action does depend on s:

$$\begin{aligned} I_{1/4\text{-BPS}} &= \frac{\pi L^2}{2G_N} \ , \\ I_{1/2\text{-BPS}} &= \frac{2\pi s^2}{G_N} \ . \end{aligned} \tag{7.32}$$

In the interest of seeing if we can bootstrap a general formula for the on-shell action with scalars turned on, having scalar generalizations of both the 1/4- and 1/2-BPS solutions would be useful.

A Useful relations

We may find it useful to compare our solutions to known solutions in minimal $\mathcal{N} = 2$ gauged supergravity. Typically, the bosonic action for this minimal supergravity is given by

$$S = -\frac{1}{16\pi G_N}\int d^4x\,\sqrt{g}\left(R - \frac{1}{4}F^{(\text{min})}_{\mu\nu}F^{\mu\nu}_{(\text{min})} + \frac{6}{L^2}\right) \ , \tag{A.1}$$

where $F^{(\min)}_{\mu\nu} \equiv \partial_\mu A^{(\min)}_\nu - \partial_\nu A^{(\min)}_\mu$. We can convert between minimal and non-minimal supergravity conventions as follows:

$$A^I = \frac{1}{\sqrt{2}} A_{(\min)} \ , \quad z^\alpha = \tilde{z}^\alpha = 0 \ . \tag{A.2}$$

For the squashed sphere solutions in particular, this means that we will set

$$e^I(r) = \frac{1}{\sqrt{2}} f(r) \ , \tag{A.3}$$

such that all of the gauge fields are proportional to the same function f. The relevant quantities for the BPS conditions are then found as follows:

$$\begin{aligned} P^3_I L^I &= -\sqrt{2} \ , \\ P^3_I e^I &= -2\sqrt{2} f \ , \\ L^I n^\pm_I &= \frac{1}{L} e^{-B-C} f' \mp \frac{2}{L^2} e^{-2A} f \ , \end{aligned} \tag{A.4}$$

where there is implicitly a complex conjugation on the n^+_I term. In turn, this means that we have the following:

$$\begin{aligned} \alpha &= MLe^{A-B}A' + Me^{C-A} - e^A - 2\zeta e^{A-B-C} f^{*\prime} + \frac{4\zeta}{L} e^{-A} f^* \ , \\ \tilde{\alpha} &= \widetilde{M}Le^{A-B}A' - \widetilde{M}e^{C-A} - e^A + 2\zeta e^{A-B-C} f' + \frac{4\zeta}{L} e^{-A} f \ , \\ \beta &= MLe^{C-B}C' - M(e^{2(C-A)} - 2) + \frac{4\zeta}{L} M f^* - e^C + 2\zeta e^{-B} f^{*\prime} - \frac{4\zeta}{L} e^{C-2A} f^* \ , \\ \tilde{\beta} &= \widetilde{M}Le^{C-B}C' + \widetilde{M}(e^{2(C-A)} - 2) - \frac{4\zeta}{L} \widetilde{M} f - e^C - 2\zeta e^{-B} f' - \frac{4\zeta}{L} e^{C-2A} f \ . \end{aligned} \tag{A.5}$$

B General squashing

The most general squashed ansatz we will consider is

$$\begin{aligned} ds^2 &= \frac{1}{4} L^2 \left(e^{2A_1(r)} \sigma_1^2 + e^{2A_2(r)} \sigma_2^2 + e^{2A_3(r)} \sigma_3^2 \right) + e^{2B(r)} dr^2 \ , \\ A^I &= e^I_1(r) \sigma_1 + e^I_2(r) \sigma_2 + e^I_3(r) \sigma_3 \ , \\ z^\alpha &= z^\alpha(r) \ , \\ \tilde{z}^\alpha &= \tilde{z}^\alpha(r) \ , \end{aligned} \tag{B.1}$$

where L is the AdS radius, the functions A_i, e^I_i, B, and the scalars are all functions only of r, and σ_i are the left-invariant $SU(2)$ Maurer-Cartan forms. The vielbein are

$$e^i = \frac{1}{2} L e^{A_i} \sigma_i \ , \quad e^4 = e^B dr \ , \tag{B.2}$$

where $i = 1, 2, 3$. The spin connections are

$$\omega^{12} = \frac{e^{2A_1} + e^{2A_2} - e^{2A_3}}{2e^{A_1 + A_2}} \sigma_3 \ , \quad \omega^{i4} = A_i' e^{-B} e^i \ , \tag{B.3}$$

with ω^{23} and ω^{31} obtained via cyclic permutations from the first expression.

From the gauge field ansatz, we find that the field strengths are given by

$$\begin{aligned} F^I &= \sum_{i=1}^{3} \left(\partial_r e_i^I dr \wedge \sigma_i + e_i^I \sigma_{i+1} \wedge \sigma_{i+2}\right) \ , \\ \tilde{F}^I &= \sum_{i=1}^{3} \left(-\frac{2}{L} e^{B+A_i-A_{i+1}-A_{i+2}} e_i^I dr \wedge \sigma_i - \frac{L}{2} e^{-B-A_i+A_{i+1}+A_{i+2}} \partial_r e_i^I \sigma_{i+1} \wedge \sigma_{i+2}\right) \ , \end{aligned} \tag{B.4}$$

where we have in mind that $A_i \equiv A_{i \bmod 3}$ and $\sigma_i \equiv \sigma_{i \bmod 3}$ for any value of $i > 3$. Note that the gauge fields automatically solve the Bianchi equations $\nabla_\mu \tilde{F}^{\mu\nu} = 0$. The Maxwell equations are a little more involved, but we end up finding the following independent equations:

$$0 = \frac{4}{L^2} e^{B+A_i-A_{i+1}-A_{i+2}} \mathcal{I}_{IJ} e_i^J - \frac{2i}{L} (\partial_r \mathcal{R}_{IJ}) e_i^J - \partial_r \left(e^{-B-A_i+A_{i+1}+A_{i+2}} \mathcal{I}_{IJ} \partial_r e_i^J\right) \ , \tag{B.5}$$

for $i = 1, 2, 3$.

We now want to evaluate the bulk on-shell action for our general squashing ansatz, and we will try to simplify the result using only the equations of motion. First, we write the on-shell action as

$$S_{\text{bulk}} = \frac{\pi L^3}{4 G_N} \int dr \ (L_1 + L_2 + L_3 + L_4 + L_5) \tag{B.6}$$

with

$$\begin{aligned} L_1 &= -\frac{\alpha}{2} R \ , \\ L_2 &= -\frac{\alpha}{4} \mathcal{I}_{IJ} F^I_{\mu\nu} F^{J\mu\nu} \ , \\ L_3 &= \frac{i\alpha}{4} \mathcal{R}_{IJ} F^I_{\mu\nu} \tilde{F}^{J\mu\nu} \ , \\ L_4 &= \alpha e^{-2B} g_{\alpha\bar{\beta}} \partial_r z^\alpha \partial_r \tilde{z}^{\bar{\beta}} \ , \\ L_5 &= \alpha g^2 \mathcal{P} \ , \end{aligned} \tag{B.7}$$

and we have defined $\alpha \equiv e^{A_1+A_2+A_3+B}$. To simplify the field strength terms, we first explicitly write them out for our ansatz:

$$\begin{aligned} \mathcal{I}_{IJ} F^I_{\mu\nu} F^{J\mu\nu} &= \mathcal{I}_{IJ} \sum_{i=1}^{3} \left[\frac{32}{L^4} e^{-2A_{i+1}-2A_{i+2}} e_i^I e_i^J + \frac{8}{L^2} e^{-2A_i-2B} \partial_r e_i^I \partial_r e_i^J\right] \ , \\ \mathcal{R}_{IJ} F^I_{\mu\nu} \tilde{F}^{J\mu\nu} &= \mathcal{R}_{IJ} \sum_{i=1}^{3} \left[-\frac{32}{L^3} e^{-A_1-A_2-A_3-B} e_i^I \partial_r e_i^J\right] \ . \end{aligned} \tag{B.8}$$

When we include the measure and all numerical factors, this means that the third term in the on-shell action (using the symmetry of the matrix $\mathcal{R}_{IJ}$) can be written as

$$\begin{aligned} L_3 &= -\frac{4i}{L^3} \sum_{i=1}^{3} \mathcal{R}_{IJ} \partial_r (e_i^I e_i^J) \\ &= -\frac{4i}{L^3} \sum_{i=1}^{3} \left[\partial_r (\mathcal{R}_{IJ} e_i^I e_i^J) - (\partial_r \mathcal{R}_{IJ}) e_i^I e_i^J\right] \ . \end{aligned} \tag{B.9}$$

By now using the Maxwell equation we can trade all instances of $\partial_r \mathcal{R}_{IJ}$ for $\mathcal{I}_{IJ}$. As a result, we find that (after a great deal of rewriting)

$$\begin{aligned} L_3 = \sum_{i=1}^{3} \Bigg[&-\frac{4i}{L^3}\partial_r(\mathcal{R}_{IJ}e_i^I e_i^J) + \frac{8}{L^4}e^{A_i-A_{i+1}-A_{i+2}+B}\mathcal{I}_{IJ}e_i^I e_i^J \\ &-\frac{2}{L^2}\partial_r(e^{-A_i+A_{i+1}+A_{i+2}-B}\mathcal{I}_{IJ}e_i^I\partial_r e_i^J) + \frac{2}{L^2}e^{-A_i+A_{i+1}+A_{i+2}-B}\mathcal{I}_{IJ}\partial_r e_i^I \partial_r e_i^J \Bigg] \ . \end{aligned} \tag{B.10}$$

Similarly, a careful rewriting of the second term yields

$$L_2 = \sum_{i=1}^{3} \left[-\frac{8}{L^4}e^{A_i-A_{i+1}-A_{i+2}+B}\mathcal{I}_{IJ}e_i^I e_i^J - \frac{2}{L^2}e^{-A_i+A_{i+1}+A_{i+2}-B}\mathcal{I}_{IJ}\partial_r e_i^I \partial_r e_i^J \right] \ . \tag{B.11}$$

Upon inspection, we find that all terms in these that are not total derivatives cancel! We are therefore left simply with

$$L_2 + L_3 = \sum_{i=1}^{3} \partial_r \left[-\frac{2}{L^2}e^{-A_i+A_{i+1}+A_{i+2}-B}\,\mathcal{I}_{IJ}e_i^I \partial_r e_i^J - \frac{4i}{L^3}\mathcal{R}_{IJ}e_i^I e_i^J \right] \ . \tag{B.12}$$

To simplify the rest of the terms in the on-shell action, we first need to study the Einstein equation. The independent degrees of freedom in the stress tensor that correspond to the Maxwell fields are captured by the following potentials:

$$\begin{aligned} V_{ij} &\equiv \mathcal{I}_{IJ}\left(4e^{2B+A_i+A_j}e_i^I e_j^J - L^2 e^{A_{i+1}+A_{i+2}+A_{j+1}+A_{j+2}}\partial_r e_i^I \partial_r e_j^J\right) \ , \\ W_{ij} &\equiv \mathcal{I}_{IJ}\left(e^{-2A_i}e_j^J\partial_r e_i^I - e^{-2A_j}e_i^I \partial_r e_j^J\right) \ . \end{aligned} \tag{B.13}$$

That is, the Maxwell portion of the stress tensor can be written entirely in terms of these potentials (as well as the metric functions). If we then study the constraints on these potentials that result from solving the Einstein equation, we find that

$$V_{ij} = W_{ij} = 0 \quad \text{for all } i \neq j \ . \tag{B.14}$$

Since W_{ij} is antisymmetric, this means all components of W_{ij} vanish. Crucially, a necessary condition for solving this equation is that at least two of the e_i^I vanish, which in turn means that $V_{ii} = 0$ for at least two different values of i. That is, it is only possible for one component on the diagonal of V_{ij} to be non-zero. However, we may also want to demand that this (potentially non-zero) component vanishes so as to make the field strength either self-dual or anti-self-dual, based on the known squashed sphere solutions.

Thus, in summary, the only way to solve the Einstein equations and also have an (anti-)self-dual field strength is to demand that

$$V_{ij} = W_{ij} = 0 \ . \tag{B.15}$$

Setting these to zero and studying the (r,r) and (ψ,ψ) components of the Einstein equation, we find that we can solve for all terms involving the scalars purely in terms of metric functions:

$$\begin{aligned}
g_{\alpha\bar{\beta}}\partial_r z^\alpha \partial_r \tilde{z}^{\bar{\beta}} &= \frac{1}{L^2}e^{2B}\left(e^{2A_1-2A_2-2A_3}+e^{-2A_1+2A_2-2A_3}+e^{-2A_1-2A_2+2A_3}\right) \\
&\quad - \frac{2}{L^2}e^{2B-2A_3} + \frac{1}{2}\left(A_1'(B'-A_1'-A_3') + A_2'(B'-A_2'-A_3') - A_1'' - A_2''\right) \ , \\
g^2\mathcal{P} &= \frac{2}{L^2}\left(e^{-2A_1-2A_2+2A_3} - e^{2A_1-2A_2-2A_3} - e^{-2A_1+2A_2-2A_3}\right) \\
&\quad + \frac{4}{L^2}e^{-2A_3} + e^{-2B}A_3'\left(B'-A_1'-A_2'-A_3'\right) - e^{-2B}A_3'' \ .
\end{aligned} \tag{B.16}$$

Note however that this does not fully solve the Einstein equations; there are still non-zero components that give constraints on the metric functions. These constraints can be solved by setting $A_1 = A_2 = A_3$, but I wonder if the constraints can more generally be solved in such a way that the on-shell action is a total derivative.

To investigate this more, we now note that

$$\begin{aligned}
L_1 + L_4 + L_5 &= \frac{2}{L^2}e^{B-A_1-A_2+A_3}\left(-e^{2A_1} - e^{2A_2} + e^{2A_3}\right) \\
&\quad + \frac{1}{2}\partial_r\left(e^{-B+A_1+A_2+A_3}(A_1' + A_2')\right) \ .
\end{aligned} \tag{B.17}$$

The (ϕ, θ) component of the Einstein equation tells us that

$$\begin{aligned}
0 &= \frac{4}{L^2}e^{B-A_1-A_2-A_3}\left(e^{2A_1} - e^{2A_2}\right)\left(-e^{4A_1} - e^{4A_2} + e^{4A_3}\right) \\
&\quad + e^{2A_1}\partial_r\left(e^{-B+A_1+A_2+A_3}(A_1' - A_3')\right) \\
&\quad - e^{2A_2}\partial_r\left(e^{-B+A_1+A_2+A_3}(A_2' - A_3')\right) \ .
\end{aligned} \tag{B.18}$$

The (θ, θ) component tells us that

$$\begin{aligned}
0 &= \frac{4}{L^2}e^{B-A_1-A_2-A_3}\left(e^{2A_2} - e^{2A_3}\right)\left(-e^{2A_1} + e^{2A_2} + e^{2A_3}\right) \\
&\quad - \partial_r\left(e^{-B+A_1+A_2+A_3}(A_2' - A_3')\right) \ .
\end{aligned} \tag{B.19}$$

Lastly, the (ϕ, ϕ) component yields

$$\begin{aligned}
0 &= \frac{4}{L^2}e^{B-A_1-A_2-A_3}\left(e^{2A_1} - e^{2A_3}\right)\left(e^{2A_1} - e^{2A_2} + e^{2A_3}\right) \\
&\quad - \partial_r\left(e^{-B+A_1+A_2+A_3}(A_1' - A_3')\right) \ .
\end{aligned} \tag{B.20}$$

Note that these equations are not all independent; the (θ, θ) and (ϕ, ϕ) constraints combine to automatically yield the (ϕ, θ) constraint. These constraints are all clearly satisfied for the round S^3 case where all A_i are equal.

Unfortunately, it doesn't seem that there is a way to apply these constraints to force the on-shell action to be a total derivative.

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