Proving the Law of the Unconscious Statistician

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Abstract

The *law of the unconscious statistician*, also referred to as the LOTUS, is a theorem that expresses the expectation value of a random variable $Z = g(X)$ that is itself a function of another random variable X in terms of the function g and the probability density function of X. It is an incredibly fundamental result in statistics that underpins the computation of almost all statistical quantities of interest. Despite its significance, though, it is often simply presented as truth with little to no discussion of why the theorem is true. Or, in the few cases where a proof is given, the proof is given only for the limiting cases of discrete random variables or invertible functions. In this note, we will present a detailed proof of the LOTUS for continuous random variables in full generality. Along the way we will also discuss the indicator function and use it to establish some useful results about expectation values.

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1 Overview

When dealing with random variables, one of the most important ways to quantify the distribution of values is to compute the expected value of the random variable. That is, if we have a continuous real-valued random variable X with a probability density function $f_X : \mathbb{R} \to [0, 1]$, then the expected value of X is given by

$$E[X] = \int dx\, x f_X(x) \ . \tag{1.1}$$

We can also take functions of random variables to obtain new random variables and study their corresponding distributions as well. In most introductory statistics resources, you will see the following generalization for the expected value of functions of random variables:

$$E[g(X)] = \int dx\, g(x) f_X(x) \ . \tag{1.2}$$

This result is usually just given and used immediately, with no explanation about where it came from. However, this is a non-trivial result, and it does not immediately follow from

(1.1). In fact, if we define a new random variable $Z = g(X)$ and then apply (1.1), we would simply obtain

$$E[g(X)] = E[Z] = \int dz\, z f_Z(z) \;, \tag{1.3}$$

which doesn't illuminate very much; we would need to figure out how to express the PDF of Z in terms of f_X and g to make a connection back to (1.2).

The identity (1.2) is sometimes referred to as the *law of the unconscious statistician*, or LOTUS for short. It is a result that we take for granted yet use in almost all statistical inference problems. In this note, I'll go through some derivations of the result.

1.1 Conventions and notation

For the rest of this note, we will only consider continuous and real-valued random variables. For any such random variable X, we will always denote its corresponding probability density function (PDF) by f_X and its cumulative distribution function (CDF) by F_X.

For any integral over the real line, we will use the implicit notation that the absence of any definite limits or integration regions means that the integral is over the entire real line, i.e.

$$\int dx\, f(x) = \int_{\mathbb{R}} dx\, f(x) = \int_{-\infty}^{\infty} dx\, f(x) \;. \tag{1.4}$$

For any random variables X and Y, we denote their joint PDF by $f_{X,Y}$ and the conditional PDF for X given Y by $f_{X|Y}$. Their corresponding CDFs are denoted by $F_{X,Y}$ and $F_{X|Y}$, respectively. Additionally, given the joint PDF, we can obtain the "marginal" PDFs for X and Y individually by integrating out the other random variable:

$$\begin{aligned} f_X(x) &= \int dx\, f_{X,Y}(x,y) \;, \\ f_Y(y) &= \int dy\, f_{X,Y}(x,y) \;. \end{aligned} \tag{1.5}$$

1.2 Assumptions

Our goal is to prove the LOTUS with as few assumptions and pre-established mathematical technology as possible. The first theorem we will make us of is *Bayes' theorem*, which relates joint, marginal, and conditional distributions:

$$f_{X,Y}(x,y) = f_{X|Y}(x\,|\,y) f_Y(y) = f_{Y|X}(y\,|\,x) f_X(x) \;. \tag{1.6}$$

The second we will make use of is the *Fubini-Tonelli theorem*, which characterizes the conditions necessary to be able to perform a double integral iteratively over one coordinate and then the other, which also allows for swapping limits of integration:

$$\iint d(x,y)\, f(x,y) = \int dx \int dy\, f(x,y) = \int dy \int dx\, f(x,y) \;. \tag{1.7}$$

Throughout this note, we will freely make use of the Fubini-Tonelli theorem without worrying about whether or not the necessary conditions are satisfied. This is slightly lazy, yes, but actually checking the conditions is a rather involved endeavor that is beyond the scope of this note.

2 Warm-up: LOTUS for invertible functions

Before moving on to the more general cases, let's first focus on the simple case where the function $g : \mathbb{R} \to \mathbb{R}$ is an invertible function, i.e. there's a unique function $g^{-1} : \mathbb{R} \to \mathbb{R}$ such that $g(x) = y$ if and only if $x = g^{-1}(y)$ for any $x, y \in \mathbb{R}$.

If we let $Z = g(X)$, then the CDF of Z can be written as:

$$F_Z(z) = P\left(Z \leq z\right) = P\left(g(X) \leq z\right) \ . \tag{2.1}$$

The invertibility of g means that the inequality $g(X) \leq z$ is satisfied if and only if $X \leq g^{-1}(z)$, and so the associated probabilities for these two inequalities must be equal. Thus, we find that the CDFs for Z and X are related as follows:

$$F_Z(z) = P\left(g(X) \leq z\right) = P\left(X \leq g^{-1}(z)\right) = F_X\left(g^{-1}(z)\right) \ . \tag{2.2}$$

The PDF for Z can thus be calculated as follows:

$$\begin{aligned} f_Z(z) &= \frac{d}{dz}\Big[F_Z(z)\Big] \\ &= \frac{d}{dz}\Big[F_X\left(g^{-1}(z)\right)\Big] \\ &= \frac{d}{dx}\Big[F_X(x)\Big]_{x=g^{-1}(z)} \frac{d}{dz}\Big[g^{-1}(z)\Big] \\ &= f_X\left(g^{-1}(z)\right)(g^{-1})'(z) \ . \end{aligned} \tag{2.3}$$

Plugging this PDF into (1.3), we find the expectation value of $Z = g(X)$ is given by

$$E[g(X)] = \int dz\, z\, f_X\left(g^{-1}(z)\right)(g^{-1})'(z) \ . \tag{2.4}$$

We now make the change of variables $z = g(x)$, or equivalently that $x = g^{-1}(z)$, which in turn means the measures dz and dx are related by

$$dx = (g^{-1})'(z)dz \ . \tag{2.5}$$

This lets us eliminate the derivative term completely from the integrand, and we are left simply with:

$$E[g(X)] = \int dx\, g(x) f_X(x) \ , \tag{2.6}$$

which is precisely the LOTUS given in (1.2).

3 Relaxing the invertibility assumption

The linchpin of the previous proof was the use of the invertibility to relate probabilities over X to probabilities over $g(X)$ by way of the following equation:

$$P\left(g(X) \leq t\right) = P\left(X \leq g^{-1}(t)\right) \ . \tag{3.1}$$

The question now is, how do we approach a proof of the LOTUS that does not use invertibility? We still need some way to relate probabilities over $g(X)$ to those over X, but it's not as straightforward as the relation above. Instead, we will have to make use of a new tool known as the *indicator function*, which we explain below.

3.1 The indicator function

Let X be a real-valued continous random variable with PDF $f_X(x)$. Then, for any $a, b \in \mathbb{R}$ such that $a < b$, the probability of X being in this range is:

$$P(a \leq X \leq b) = \int_a^b dx\, f_X(x) \ . \tag{3.2}$$

More generally, for any measurable set $A \subseteq \mathbb{R}$, the probability of X being in that set is given by

$$P(X \in A) = \int_A dx\, f_X(x) \ . \tag{3.3}$$

Notice that we specified that the set must be *measurable*, so that the set has an appropriate Lebesgue measure that makes the integral (3.3) well-defined. We'll discuss this more in Section 3.3.

Now, we define the indicator function $\mathbf{1}_A : \mathbb{R} \to \{0, 1\}$ such that

$$\mathbf{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \ . \tag{3.4}$$

We can use this function to turn the integral over A into an integral over all of $\mathbb{R}$:

$$\begin{aligned} P(X \in A) &= \int_A dx\, f_X(x) \\ &= \int_A dx\, (1 \cdot f_X(x)) + \int_{\mathbb{R}\setminus A} dx\, (0 \cdot f_X(x)) \\ &= \int_A dx\, \mathbf{1}_A(x) \cdot f_X(x) + \int_{\mathbb{R}\setminus A} dx\, \mathbf{1}_A(x) \cdot f_X(x) \\ &= \int dx\, \mathbf{1}_A(x) f_X(x) \ , \end{aligned} \tag{3.5}$$

where the absence of limits should be understood as integrating over the entire real line.

Given the function $g : \mathbb{R} \to \mathbb{R}$, we can now define, for any value of t, the following set:

$$A_t = \{x \in \mathbb{R} : g(x) > t\} \ . \tag{3.6}$$

That is, the elements in A_t comprise all real numbers that satisfy $g(x) > t$. Therefore, the inequality $g(x) > t$ is satisfied if and only if $x \in A_t$. We can thus write the probability of satisfying this inequality as follows:

$$\boxed{P(g(X) > t) = P(X \in A_t) = \int dx\, \mathbf{1}_{A_t}(x) f_X(x)} \ . \tag{3.7}$$

This crucial identity relates cumulative probabilities over $g(X)$ to an integral over the PDF of X, albeit convolved with the indicator function. This is the more general version of the invertibility relation (3.1), and in fact it reduces to that relation in the special case where g is invertible.

3.2 Relating expectation values to CDFs

We now need to find a way to relate expectation values to CDFs, so that (when combined with (3.7)) yields a direct relationship between an expectation value of $g(X)$ to an integral over f_X. First, as a prelude, suppose we have a non-negative real number $x \in \mathbb{R}^+$. We can trivially represent this number as an integral as follows:

$$x = \int_0^x dt \ . \tag{3.8}$$

Then, by defining the set $S_x = \{t \in \mathbb{R}^+ : t < x\}$, we can express this as an integral over the indicator function:

$$x = \left(\int_0^x dt \cdot 1\right) + \left(\int_x^\infty dt \cdot 0\right) = \int dt \, \mathbf{1}_{S_x}(t) \ . \tag{3.9}$$

By replacing x with a real-valued non-negative random variable X, we obtain the following:

$$X = \int dt \, \mathbf{1}_{S_X}(t) \ , \quad S_X = \{t \in \mathbb{R} : t < X\} \ . \tag{3.10}$$

We now take the expectation value of both sides of this equation. Using the Fubini-Tonelli theorem to bring the expectation value under the integral sign, we find that:

$$E[X] = E\left[\int dt \, \mathbf{1}_{S_X}(t)\right] = \int dt \, E\left[\mathbf{1}_{S_X}(t)\right] \ . \tag{3.11}$$

To proceed, we now need to understand the expectation value of the indicator function in the integrand. First recall that, for any discrete random variable Y whose possible outcomes are denoted by the discrete set Ω, the expectation value of that discrete random variable is simply

$$E[Y] = \sum_{\omega \in \Omega} \omega \, P(\omega) \ , \tag{3.12}$$

where $P(\omega)$ is the probability associated with the outcome ω, with normalization such that $\sum_{\omega \in \Omega} P(\omega) = 1$.

Now, consider the indicator function $\mathbf{1}_{S_X}(t)$. For any fixed value of t, this is a random variable that depends on X. Moreover, it is a *discrete* random variable, because the values it can take are 1 (which occurs when $t \in S_X$) and 0 (which occurs when $t \notin S_X$). Thus, the expectation value of this indicator function can be written as:

$$\begin{aligned} E[\mathbf{1}_{S_X}(t)] &= 1 \cdot P(t \in S_X) + 0 \cdot P(t \notin S_X) \\ &= P(t \in S_X) \\ &= P(X > t) \ . \end{aligned} \tag{3.13}$$

That is, the expectation value of the indicator function is simply the probability that the indicator condition is satisfied. Using this result in (3.11), we find that

$$\boxed{E[X] = \int dt \, P(X > t)} \ , \tag{3.14}$$

for any non-negative random variable X.

3.3 Caveat: measurability

The one caveat to all of this is that A_t must be a *measurable set*, which (loosely speaking) means that one can assign a Lebesgue measure to the set that allows the computation of integrals like (3.7). The real numbers $\mathbb{R}$ are themselves a measurable set, with the Lebesgue measure being the standard one that measures intervals by their Cartesian length. However, the set A_t (defined as all real numbers x that satisfy $g(x) > t$) will not be measurable for all choices of g. In fact, A_t will be measurable if and only if g is a *measurable function*, which (again, loosely speaking) is one that maps between measurable spaces and preserves the structure of the spaces such that the preimage of any measurable set is also measurable.

The condition that g be measurable is a less stringent condition than invertibility. In fact, measurability is a strict generalization of invertibility; any invertible function is necessarily measurable as well. There are far more measurable functions that are not invertible, though, so this generalization is a pretty big win for us.

For example, consider the function $g(x) = x^2$. When acting on the domain $\mathbb{R}^+$, this function can be inverted, but when acting on the entire real line the function is no longer invertible. So, if we have a real-valued random variable X that takes both positive and negative values, we may be interested in computing the expectation value $E[X^2]$, and the lack of invertibility means that we can't simply use the result from Section 2. Hope is not lost, though: we'll prove in the next section that the LOTUS holds for general measurable functions, which will let us compute $E[X^2]$ after all.

4 LOTUS for measurable functions

Through careful use of the indicator function, we've arrived at two fundamental relations (3.7) and (3.14) that relate expectation values of functions of random variables to integrals of CDFs and PDFs without having to appeal to invertibility. We will now make use of these relations to prove that the LOTUS holds for arbitrary measurable functions. We will first focus on the case of a non-negative function for simplicity, before moving on to the more general case.

4.1 LOTUS for non-negative measurable functions

First, we consider a non-negative measurable function $g : \mathbb{R} \to \mathbb{R}^+$. The non-negative assumption on g means that for any random variable X, the random variable $g(X)$ will be non-negative, and thus we can apply (3.14):

$$E[g(X)] = \int_0^\infty dt\, P(g(X) > t) \ . \tag{4.1}$$

Then, we denote $A_t = \{x \in \mathbb{R} : g(x) > t\}$ as the set of all real values where the inequality inside the integrand is satisfied. Since we have assumed g to be measurable, we can use (3.7) to rewrite the integrand as

$$P(g(X) > t) = \int dx\, \mathbf{1}_{A_t}(x) f_X(x) \ . \tag{4.2}$$

Putting that back into the integral, the expectation value is given by

$$E[g(X)] = \int_0^\infty dt \int dx \, \mathbf{1}_{A_t}(x) f_X(x) = \int dx \, f_X(x) \int_0^\infty dt \, \mathbf{1}_{A_t}(x) \ , \tag{4.3}$$

where the second equality comes from applying the Fubini-Tonelli theorem to swap the orders of integration.

Note that the indicator function $\mathbf{1}_{A_t}(x)$ will be one only when $0 \leq t < g(x)$ and zero otherwise, and so the integral on the right-hand side can be rewritten as

$$E[g(X)] = \int dx \, f_X(x) \int_0^{g(x)} dt = \int dx \, f_X(x) g(x) \ . \tag{4.4}$$

We have thus arrived at the LOTUS (1.2) for any non-negative and measurable function g.

4.2 LOTUS for general measurable functions

We are now equipped to prove the LOTUS in the general case where g is a real-valued measurable function $g : \mathbb{R} \to \mathbb{R}$ that can take any value. To proceed, we first break g up into its positive and negative pieces as follows:

$$g(x) = g^+(x) - g^-(x) \ , \tag{4.5}$$

where we have defined $g^\pm$ as follows:

$$g^+(x) = \begin{cases} g(x) & g(x) > 0 \\ 0 & g(x) < 0 \end{cases} \tag{4.6}$$

$$g^-(x) = \begin{cases} 0 & g(x) > 0 \\ -g(x) & g(x) < 0 \end{cases} \tag{4.7}$$

Importantly, both g^+ and g^- are non-negative functions, i.e. $g^\pm : \mathbb{R} \to \mathbb{R}^+$. And, since g is measurable, $g^\pm$ will clearly be measurable as well, and thus we can apply the results from the previous section for each function individually. We therefore find that the LOTUS applies for the positive and negative parts of g individually:

$$E\left[g^\pm(X)\right] = \int dx \, f_X(x) g^\pm(x) \ , \tag{4.8}$$

for any random variable X.

Using the additive property of the expectation value (see Appendix A), we now derive the following:

$$\begin{aligned} E[g(X)] &= E\left[g^+(X) - g^-(X)\right] \\ &= E\left[g^+(X)\right] - E\left[g^-(X)\right] \\ &= \left[\int dx \, f_X(x) g^+(x)\right] - \left[\int dx \, f_X(x) g^-(x)\right] \\ &= \int dx \, f_X(x) \left(g^+(x) - g^-(x)\right) \\ &= \int dx \, f_X(x) g(x) \ . \end{aligned} \tag{4.9}$$

Once again, we have arrived at the LOTUS. Moreover, the only assumption we have put on g is that it is measurable, and so this constitutes the most general version of the LOTUS so far. In fact, it is the most general version that exists for real-valued random variables; unmeasurable functions of random variables are not actually random variables themselves, from a probability theory point of view.

4.3 Extensions, other cases, and quibbles

We have so far only worked with continuous real-valued random variables, but it should come as no surprise that the LOTUS also holds for continuous random variables on other spaces, though one has to be less cavalier than we have here about the validity of various integral manipulations.

The LOTUS also applies to discrete random variables. For a discrete random variable X and some function g that acts on the set of values of X, the discrete LOTUS is obtained by replacing integrals with sums:

$$E[g(X)] = \sum_i g(x_i) f_X(x_i) \ , \tag{4.10}$$

where i indexes the possible values x_i that X can take. This discrete version is automatically implied by the continuous version. Moreover, we no longer have to worry about measurability of g; its image is a countable set and so the discrete analogs of the manipulations performed in Section 3 follow automatically.

It's also worth noting that the generalized version of the LOTUS for multiple random variables, i.e.

$$E[g(X_1, \ldots, X_n)] = \int d^n x \, g(x_1, \ldots, x_n) f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) \ , \tag{4.11}$$

immediately follows from the proof here by simply replacing all occurences of the PDF f_X with the joint PDF $f_{X_1,\ldots,X_n}$ and continuing to assume that the Fubini-Tominelli theorem introduces no extra complications whatsoever.

The last thing we will mention is that we have not considered the issue of *convergence rates* or *divergences* anywhere in this note. We have assumed that all integrals are finite and computable, but this is only true if the integrand has sufficiently well-behaved asymptotics and is free of non-integrable singularities. In general, one should consider the specifics of the random variable X and function g to ensure that all of the steps taken in this proof are valid. This goes way too far into advanced topics of real analysis and measure theory for my liking, though, and I recommend finding contentment with the fairly minimal set of assumptions used in this note.

Appendix

A Additivity of expectation values

The last piece of the puzzle we need is to back up our claim that the expectation value is additive, i.e. $E[X+Y] = E[X] + E[Y]$ for any random variables X, Y, which we used in equation (4.9) to prove the LOTUS for measurable functions.

Additivity is usually presented as an immediately obvious property of the expectation value, but I find that this is only "obvious" if we *assume the LOTUS to be true*! So, our goal in this section is to go the opposite direction and derive this additivity property without appealing to LOTUS anywhere, so that we can then use this in the previous section to prove the LOTUS.

Let X and Y be real-valued continuous random variables, with no assumptions made on whether or not they are independent. We can take their sum $X+Y$ to obtain a new random variable whose expectation value is given by

$$E[X+Y] = \int dt\, t\, f_{X+Y}(t) \; . \tag{A.1}$$

To proceed, we turn this into a joint PDF over both $X+Y$ and Y by using (1.5) and then apply Bayes' theorem to turn this joint PDF into a conditional PDF:

$$\begin{aligned} f_{X+Y}(t) &= \int dy\, f_{X+Y,Y}(t,y) \\ &= \int dy\, f_{X+Y|Y}(t\,|\,y) f_Y(y) \; . \end{aligned} \tag{A.2}$$

Our goal now is to relate the conditional PDF $f_{X+Y|Y}$ to the slightly simpler conditional PDF $f_{X|Y}$. To do this, we consider the conditional CDF $F_{X+Y|Y}$ and write it as

$$F_{X+Y|Y}(t\,|\,y) = P(X+Y \le t\,|\,Y=y) \; . \tag{A.3}$$

Of course, the probability $P(X+Y \le t)$ is precisely the probability $P(X \le t-Y)$. And, when we have the extra conditional assumption that $Y=y$, we are free to replace Y with y in this probability at no cost. We therefore find that

$$\begin{aligned} F_{X+Y|Y}(t\,|\,y) &= P(X \le t-Y\,|\,Y=y) \\ &= P(X \le t-y\,|\,Y=y) \\ &= F_{X|Y}(t-y\,|\,y) \; . \end{aligned} \tag{A.4}$$

Treating the value of y as a constant and differentiating both sides with respect to t, we find that the corresponding PDFs satisfy the same relation:

$$f_{X+Y|Y}(t\,|\,y) = f_{X|Y}(t-y\,|\,y) \; . \tag{A.5}$$

Putting this all together, we find the expectation value can be written as

$$E[X+Y] = \int dt \int dy\, t\, f_{X|Y}(t-y\,|\,y) f_Y(y) = \int dt \int dy\, t\, f_{X,Y}(t-y,y) \; , \tag{A.6}$$

where the second equality came from applying Bayes' Theorem again. We now go once more unto the breach and apply the Fubini-Tonelli theorem to switch the order of integration and first do integration over t for fixed y. By making the change of variables $t = x + y$, the expression becomes:

$$E[X+Y] = \int dy \int dx\, (x+y) f_{X,Y}(x,y)\ . \tag{A.7}$$

We're almost done: we can break this integral of a sum into a sum of integrals, swap integration orders when appropriate, and use (1.5) to turn each of these integrals over joint PDFs into integrals over a single PDF each:

$$\begin{aligned} E[X+Y] &= \int dx \int dy\, x f_{X,Y}(x,y) + \int dy \int dx\, y f_{X,Y}(x,y) \\ &= \int dx\, x f_X(x) + \int dy\, y f_Y(y) \\ &= E[X] + E[Y]\ . \end{aligned} \tag{A.8}$$

Thus we've arrived at the desired conclusion.

Something worth reiterating is that this is a very long-winded proof compared to the ones usually used to establish additivity. These more standard proofs jump over most of these steps by appealing to the LOTUS from the get-go and simply writing that

$$E[X+Y] = \int dx \int dy\, (x+y) f_{X,Y}(x,y)\ . \tag{A.9}$$

We didn't have the luxury of going straight to this step, and instead had to wind our way through various iterations of Bayes' theorem to arrive at the same conclusion. This is what allows us to use this property in proving the LOTUS, rather than going the other way around.