General Relativity 2020-21

Homework 2

Due before the lecture on December 15 2020

Problem 1

Consider Einstein's equation in vacuum with a non-zero cosmological constant Λ (work in four-dimensional space-time)

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \, .$$

1. Show that the Einstein equations are equivalent to the following set of equations
$$R_{\mu\nu} = c\Lambda g_{\mu\nu} \, .$$
Find the real constant c.

2. Solve for the most general spherically symmetric metric (in the same coordinates (t, r, θ, ϕ) we used for the Schwarzschild solution) that reduces to the ordinary Schwarzschild solution for $\Lambda = 0$.[1]

3. Write down the equation of motion for radial geodesics for your metric with $\Lambda \neq 0$ in terms of an effective potential (just like this was done in the case $\Lambda = 0$ in Equation (5.66) of Carroll's book). Sketch the effective potential for massive particles and *briefly* compare your results for $\Lambda > 0$ and $\Lambda < 0$ with the ones for vanishing Λ.

Problem 2

In the presence of an electromagnetic field and in a curved space-time a particle of mass m and charge q obeys the following modification of the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{q}{m} F^\mu{}_\nu \frac{dx^\nu}{d\tau} \, .$$

Here $\Gamma^\mu_{\rho\sigma}$ is the Christoffel symbol for the metric of the background space-time and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ is the elctro-magnetic Maxwell tensor associated with the Maxwell field A_μ. Assume that the particle is moving in the background of a Reissner-Nordström black hole with metric

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \, , \qquad f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2} \, ,$$

and Maxwell field

$$A = \frac{Q}{r} dt \, .$$

Using two of the Killing vectors of the metric find explicit expressions for the conserved energy and the magnitude of the angular momentum for the moving particle. Discuss your answer and compare it with the expressions for the Schwarzschild solution studied in class.

[1] Hint: Please, do not calculate the Ricci tensor explicitly if you can avoid that. You are allowed to use the explicit expressions in Chapters 5.1 and 5.2 of Carroll's book but you have to refer to the formulae you use.