General Relativity 2020-21

Homework 1

Due before the lecture on November 10 2020

Problem 1

Consider the following metric on the three-sphere of unit radius

$$\mathrm{d}s^2 = \frac{1}{4}\left[\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 + (\mathrm{d}\psi + \cos\theta \mathrm{d}\phi)^2\right] \ .$$

1. Calculate the Christoffel symbol, $\Gamma^{\mu}_{\nu\sigma}$, for this metric.
2. Calculate the Riemann tensor, Ricci tensor, and Ricci scalar.[1]
3. Show that the following two identities are obeyed

$$R_{\mu\nu\rho\sigma} = c_1(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma}) \ , \qquad\qquad R_{\mu\nu} = c_2 g_{\mu\nu} \ ,$$

   and find the real constants c_1 and c_2.
4. Use that the angles have the following ranges $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$, $\psi \in [0, 4\pi]$ to compute the volume of the sphere

$$V = \int \sqrt{g} d^3x \, ,$$

   where $\sqrt{g}$ is the determinant of the metric.

Problem 2

General Relativity finds important applications in GPS navigation. Clocks at different heights tick at a different pace and ignoring these corrections would affect timing comparison between the surface of the Earth and satellites in orbit around it. This in turn will lead to large cumulative distance errors. To understand this better we will use the following metric which is a good approximation away from the surface of the Earth[2]

$$\mathrm{d}s^2 = -(1 + 2\Phi)c^2\mathrm{d}t^2 + (1 - 2\Phi)\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \ ,$$

where r is a coordinate that starts from the center of the Earth at $r = 0$ and

$$\Phi = -\frac{GM}{c^2 r} \ .$$

Here G is Newtown's constant and M is the mass of the Earth.

[1] Tip: To simplify your calculation use the symmetries of the Riemann tensor and compute only the independent components.

[2] Note that here I have restored the speed of light c in order to estimate the results in this problem in the usual SI units.

1. Show that a satellite in circular motion at fixed $r = R$ and $\theta = \pi/2$ obeys

$$\dot{t}^2 = \frac{R^3}{GM}\dot{\varphi}^2 ,$$

where the dot denotes derivative with respect to the proper time.

2. How much proper time elapses while a satellite at distance $R_o = 20200$ km from the surface of the Earth goes through one orbit (see `https://en.wikipedia.org/wiki/Medium_Earth_orbit` for some context). You can assume that the Earth is a sphere of radius approximately $R_E = 6370$ km.

3. How does your result in 2. above compare with the proper time elapsed on a stationary clock on the surface of the Earth?

4. Estimate the error you will have in this calculation for one orbit if you have ignored the effects of General Relativity.